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Keyword: many-to-one matching, equality of opportunity, rotation, stability

JEL Cassification: C78, D63

Equal opportunities in many-to-one matching markets*

Domenico Moramarco[†‡] Umutcan Salman [§]

June 2023

Abstract

We introduce a notion of fairness, inspired by the equality of opportunity literature, into many-to-one matching markets endowed with a measure of the quality of a match between two entities in the market. In this framework, fairness considerations are made by a social evaluator based on the match quality distribution. We impose the standard notion of stability as minimal desideratum and study matching that satisfy our notion of fairness and a notion of efficiency based on aggregate match quality. To overcome some of the identified incompatibilities, we propose two alternative approaches. The first one is a linear programming solution to maximize fairness under stability constraints. The second approach weakens fairness and efficiency to define a class of opportunity egalitarian social welfare functions that evaluate stable matchings. We then describe an algorithm to find the stable matching that maximizes social welfare.

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1 Introduction

Economists are often concerned with allocation problems and search for mechanisms that realize desirable allocations. Many of such problems can be framed as matching problems in contexts where the price mechanism fails to adequately match demand and supply. Classic examples are the marriage market or the allocation of students to public schools. In this paper, we focus on the latter as a prominent example of many-to-one matching settings, to study allocation mechanisms that equalize outcome opportunities for entities in one side of the market.

Many political philosophers (Rawls, 1971; Sen *et al.*, 1980; Dworkin, 1981a,b) have debated whether all inequalities should be considered unacceptable or if there are certain inequalities that a fair society should tolerate and preserve. Recently, economists

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(Cappelen *et al.*, 2007, 2013; Alesina *et al.*, 2017) have also investigated preferences for redistribution and tried to identify inequalities that individuals consider unfair. All these scholars converge on the idea that inequalities stemming from individual characteristics out of control or responsibility are unfair and detrimental to socio-economic development. Consequently, a fair society- one that realizes equality of opportunity (EOp) - should aim at levelling the playing field so that the final outcome of each individual is ultimately due to their own choices. This paper focuses on the prominent interpretations of the EOp paradigm (see Ramos & Van de Gaer, 2016, for a survey), which defines social justice as equality of expected outcomes across groups of individuals with similar circumstances (such as, for example, gender, parents' education, and ethnicity) out of their control.

Roemer & Ünveren (2017) argue that a key source of observed inequality of opportunity (IOp) in income is the education premium that some students gain from attending better schools. Other researchers, such as Corak (2013), have emphasized the importance of primary and secondary education in shaping opportunity and social mobility. The school allocation setting is therefore of great interest for the EOp literature and, for ease of exposition, the focus of our paper. It will be clear, however, that our framework and results can be extended to any instance of many-to-one matching markers.

In many countries (e.g. Sweden, Chile, France, Turkey and the Netherlands) and cities (e.g. New Orleans, New York, and Boston) around the globe, students are assigned to public schools through centralized school choice systems. A school choice system is a two-sided matching market in which there are students (or their parents) on one side of the market, and public schools on the other side of the market. In this system, students (or their parents) submit to the educational authority their preferences for public schools, and the schools rank students based on certain criteria. With no (or fixed) tuition fees for public schools, the central authority uses an algorithm to match students with schools while respecting preferences and priorities. The goal of the educational authority is to design an algorithm that finds assignments with desirable properties such as stability and Pareto efficiency. Stability is the central fairness notion in two-sided matching markets; it requires that in the final assignment there exist no student-school pair that would prefer each other to their current matching. Pareto efficiency can be regarded as a welfare concept in matching markets; an assignment is said to be Pareto efficient if there is no other assignment that makes a student better off without hurting some other student. The incompatibility between these properties is well-established in the literature (see Abdulkadiroğlu & Sönmez, 2003) and a well known (compromise) solution to this impossibility is the Deferred Acceptance (DA) algorithm (Gale & Shapley, 1962) used, for example, by the educational authorities in New York City to assign students to public high schools.

To promote disadvantaged groups of students or to reduce school segregation, the matching literature has proposed various affirmative action policies. The three main ones are: majority quotas (Abdulkadiroğlu & Sönmez, 2003), minority reserves (Hafalir *et al.*, 2013), and priority-based affirmative action. Majority quotas limit the number of majority students that can be admitted to a school, minority reserves retain seats for minority students, and priority-based affirmative action prioritizes minority students in the admissions process. While it has been shown (see Kojima, 2012; Doğan, 2016; Afacan & Salman, 2016) that these policies may have negative effects on the targeted groups, this paper argues that focusing on few particular minorities may lead us to neglect other disadvantaged groups. For example, Harvard's admission policy has been criticized as one that indirectly favours Afro-America applicants at the expenses of the Asian ones.[1] Although we do not claim here that Harvard implements an affirmative action policy in favour of Afro-America applicants, it is indeed possible for such a policy to harm other disadvantaged groups. Intuitively, this happens because the focus on improving the outcome of a single disadvantaged group, typical of affirmative action policies, imposes no restrictions on who has to loose for this to happen (in Appendix A we provide an example of this). This paper aims to implement a more holistic approach, typical of the EOp paradigm, which takes into account all groups simultaneously.

Abdulkadiroğlu *et al.* (2020) find that parent's preferences do not always align with the quality of education their children can receive from given schools. Therefore, in this paper, we follow Abdulkadiroglu *et al.* (2021) in considering the quality of a student-school match as the relevant outcome for the central authority to evaluate assignments. More precisely, we consider match quality to be the potential educational outcome a student can receive by attending a school. Although we will discuss particular cases, match quality is assumed to be measured by a central authority for any possible student-school pair. Under such assumption, equalizing educational opportunities for students coincides with equalizing opportunities for match quality. There is a clear parallel between the school choice setting discussed so far and other many-to-one matching problems. An example is the hospital-residency problem where we have a fixed number of hospitals or medical programs, each with a limited number of available residency positions, and a group of medical students who are seeking to be matched with a hospital program for their residency training. Other examples are the adoption problem - where many prospective parents may apply to adopt a child, but there are a limited number of children available for adoption - or the refugees allocation - where refugees apply for different reception centres with limited capacity. In all these settings, partial information (for example, on the characteristics of the hospitals) or the simple dis-alignment between the preferences of the evaluator and those of the entities in the market, can justify the focus on match quality as a the relevant outcome: one which the evaluator wants to equalize opportunities for.

In this paper, we consider the standard school choice setting, augmented with the partition of students into types (groups of individuals with the same circumstances) and a measure of match quality, and we consider three desirable properties for a school assignment: stability, efficiency and fairness. While the former corresponds to the standard requirement in matching theory, the second property has already been proposed by

[1] See also Students for Fair Admissions, Inc. v. President and Fellows of Harvard College (Docket 20–1199).

Abdulkadiroglu *et al.* (2021). The fairness requirement is, however, new and is based on the idea of minimizing the inequality in types' expected educational outcome. We highlight the incompatibility between these three requirements and provide two approaches to solve it. The first solution consists in using linear programming to identify the fair, or the efficient, allocation within the set of stable ones. The second approach is a normative one which defines a family of opportunity egalitarian social welfare functions, in line with Peragine (2004), and an algorithm - the Stable Opportunity Egalitarian (SOE) - based on Cheng *et al.* (2008), which maximize social welfare within the set of stable matchings.

We contribute to the literature in three ways. First, to the best of our knowledge, this is the first paper that introduces the concept of Equality of Opportunity à la Roemer (1998) in many-to-one matching settings. Second, despite the well-known and documented importance of public education in enhancing opportunities, the EOp literature has hardly included centralized school admission systems within the set of policy recommendations. This paper shows the possibility of designing school allocation mechanism inspired to opportunity egalitarian fairness principles. Finally, our use of Cheng *et al.* (2008)'s proposal in the SOE algorithm shows how this procedure can be implemented in other many-to-one matching settings to maximize utilitarian social welfare functions over the set of stable matchings.

The paper is organized as follows. Section 2 introduces the basic notation and definitions. Section 3 introduces the desirable properties, discusses their compatibility and the complexity of optimizing over the set of stable matchings. Section 4 introduces a family of opportunity egalitarian social welfare functions and the SOE algorithm. Section 5 discusses the positive and normative implication of the algorithm. Section 6 concludes.

2 The framework

An instance of centralized school choice problem, with EOp components, is a tuple $\mathcal{I} = \langle I, S, P, \succ, q, T, \trianglelefteq, U\rangle$, where $I = \{i_1, i_2, ..., i_{|I|}\}$ is the set of students and $S = \{s_1, s_2, ..., s_{|S|}\}$ is the set of schools. We denote P= $(P_{i_1}, ..., P_{i_{|I|}})$ the students' preferences profile such that, for all $i \in I$ and for all $s, s' \in S$, $s \; P_i \; s'$ means that i strictly prefers s to s'. We assume preference profiles to be complete and strictly linear. The school's priority profile is $\succ= (\succ_{s_1}, ..., \succ_{s_{|S|}})$; for all $s \in S$, $\succ_s$ is the complete and strictly linear priority ranking of school $s \in S$ over I, so that $i \succ_s i'$ means that i has higher priority than i' of being admitted to s. The vector $q = (q_{s_1}, ..., q_{s_{|S|}})$ is the quota profile of schools, so that each school $s \in S$ can admit at most $q_s \in \mathbb{N}_{++}$ students.

We assume that the population of students can be partitioned in mutually exclusive subgroups which, following the EOp literature,(Roemer, 1998) we call types. We denote $T = \{t_1, t_2, ..., t_{|T|}\}$ the set of types (or type partition). As in many applications (Atkinson & Bourguignon, 1982; Peragine, 2002, 2004), we assume the existence of a complete and transitive pre-order of types such that $t \trianglelefteq t'$ means that students of type $t \in T$ are not more advantaged (or have weakly higher needs) than those of $t' \in T$.

The last key ingredient is the educational outcome or match quality. We assume that there exists an $|I| \times |S|$ match quality matrix U, such that each cell (i, s) of U represents the potential[2] educational outcome of student i from attending school s. We assume match quality to be comparable across students and schools. For convenience, abusing notation, we will sometimes refer to a match quality function $U : I \times S \to \mathbb{N}_{++}$, whose value $U(i, s)$ is the match quality of the student-school pair (i, s), which coincides with the relative entry of the match quality matrix. We assume U to be exogenously given by a central authority or an evaluator with sufficient information to assess potential educational outcomes.

Our assumption does not exclude relevant ways of measuring match quality. The first approach consists in measuring potential educational outcomes according to individual preferences, so that the better preferred school provides higher match quality than a less preferred one. Such a measure can rely on the assumption that attending the favourite school boosts motivation and potential outcomes of students, as well as on the idea that students (or parents) have sufficient information to assess schools' quality. As we will discuss afterwards, from a normative perspective, such a match quality measure is in line with the opportunity egalitarian principle of holding individual responsible for their preferences.

A second approach to assess match quality relies on schools' priority rankings; these are often based on previous results or test scores that can be indicative of the potential educational outcome. At the same time, schools may have a better understanding, based on past experience, of the way students with different abilities respond to their particular teaching methods. These two criteria for assessing match quality have, however, their drawbacks: students (or parents) may not have full knowledge of the school characteristics; students' preferences may be influenced by external factors that are not relevant (or can be detrimental) for educational outcome;[3] schools may have incentives to admit students with better parental background or prefer a certain student composition in order to preserve a status.

A third way of defining match quality consists in taking the average income, or higher education achievement, of other students that attended a given school. For example, one may have information on the income of workers, form a particular ethnic group, that attended school s in the previous years and use it to define the match quality of future students of s belonging to the same ethnic group.

Another criterion to define match quality may rely on rankings by independent authorities or organization. For example, a quick search on the net can provide future university students with the top-ranked universities and departments worldwide. Similar rankings are likely to exist for smaller geographical areas and other education degrees, so that the educational authority can assign match quality 1 to the worst school in the

[2]The outcome is potential because we do not assume to observe student's future effort once enrolled.

[3]For example, a student may prefer the closest school even if it has low quality, rather than a better school at five minutes walking distance.

area, 2 to the second worst, 3 to the next one and so on. This is also in line with the idea that two students attending the same school would get the same educational outcome.

A centralized school choice setting is a two-sided matching market where there is no price mechanism to clear the market. Hence, an educational authority must apply a matching algorithm to assign students to the available slots in public schools. The result of such an algorithm is an *assignment*,[4] which we express as a function μ from the set $I \cup S$ into the power set of $I \cup S$ such that: *(i)* for all $i \in I$, $\mu(i) \in S$; *(ii)* for all $s \in S$, $|\mu(s)| \leq q_s$ and $\mu(s) \subseteq I$; *(iii)* $\mu(i) = s$ if and only if $i \in \mu(s)$. In words, $\mu(i) = s$ means that student i is enrolled to school s, and $\mu(s)$ denotes the subset of students admitted to school s. A student i is assigned to a school s if and only if student i is one of the students that school s admits. We assume that, in any final assignment, each student is assigned to a school and all the students are entitled to a seat in any schools.

As in Cheng *et al.* (2008), we assume match quality to satisfy the following *independence property.*

Assumption 1 (IND)**.** For all $i \in I$ and $s \in S$, the match quality $U(i, s)$ is function of i and s alone.

A key implication of IND is that $U(i, \mu(i))$ does not depend on the assignment of any other student $j \in I/\{i\}$. This makes it difficult to account for peer effects when measuring match quality. It is worth underlining though that one can still define a match quality measure that accounts for all the peer effects stemming from interactions with other cohorts' students already enrolled, without violating IND. To put it differently, IND requires match quality to depend only on the characteristics of student and school *before* to the matching. Therefore, the quality of future matches can be influenced by the characteristics of the current student body.

Similar reasoning concerns the issue of school segregation. As it will be clearer later, depending on the match quality matrix, school segregation may be compatible with equality of opportunity. This is, however, only in part due to our assumption. While IND prevents us from adapting match quality to the demographics of students admitted simultaneously, the ethnic composition of the previously admitted students can be taken into account by the central authority that assesses match quality.

3 Desirable properties

In this section we introduce desirable requirements for student-school assignments, discuss their compatibility and propose a first approach to realizing equality of opportunity in a school choice setting.

It is standard in the matching literature to require stability of the final allocation. Let $\mathcal{M}$ denote the set of all potential matching for the an instance $\mathcal{I}$ at hand.[5] A stable matching μ, in a two sided matching market, can be formalized as follows.

[4] Throughout the text we will use the terms assignment, matching and allocation as synonymous.
[5] We omit the reference to $\mathcal{I}$ in $\mathcal{M}$ to maintain a lighter notation.

Definition 1 (Stability)**.** The matching $\mu \in \mathcal{M}$ is stable if there is no student-school pair $(i, s) \in I \times S$ such that $s\ P_i\ \mu(i)$ and either $|\mu(s)| < q_s$ or $|\mu(s)| = q_s$ and $i \succ_s i'$ for some $i' \in \mu(s)$. The subset $M \subseteq \mathcal{M}$ is the set of stable matchings.

In words, an assignment μ is stable if there is no student-school (i, s) pair such that student i prefers school s to his assignment and either school s has an empty slot or school s has no available slot and it ranks i higher than some of its admitted student. Gale & Shapley (1962) prove that, for any instance $\mathcal{I}$, there always exist a stable assignment which can be obtained using the Deferred Acceptance algorithm (see Section 4.1 for details). Hence, $M \neq \emptyset$

Stability guarantees the respect for boteh schools' priorities and student's preferences. In a non-stable assignment there exist a student i that prefers school s to her current assignment and has higher priority than a student j admitted to s. This situation, which would be deemed unfair in the social choice literature, gives student i the right to object the final assignment and complaint to the public authority. This type of complaints can be a burden for the authorities which may find stability to be desirable also from a practical point of view. In line with the prominent approaches in the literature, we keep stability as the necessary requirement for any desirable school allocation.

Let $\mu \in M$ denote a potential assignment. The rest of this section discusses desirable properties of μ. As it will be clearer afterwards, these properties are concerned with the distribution and sum of match qualities induced by μ. Consequently, in $\mathcal{M}$, they clash with stability, which considers preferences and priority rankings independently of match quality. This motivates our focus on desirable properties for stable matchings alone.

As also argued by Abdulkadiroglu *et al.* (2021), it is desirable to obtain an assignment $\mu \in M$ that maximized aggregate match quality. This is a natural efficiency requirement that is supported both within and beyond the fairness literature. If U is the potential educational outcome, most utilitarian social planners would aim to maximize the sum of educational outcomes, as this increases human capital accumulation, growth, and social welfare. The following axiom formalizes this requirement.

Axiom 1 (Strong Efficiency)**.** There is no other assignment $\mu' \in M$ such that $\sum_{i \in I} U\left(i, \mu'(i)\right)$ is strictly greater.

The next property is a fairness requirement inspired to the EOp paradigm. For any match quality measure U and assignment μ, let

$$\bar{u}\left(t, \mu\right) = \frac{1}{|t|} \sum_{i \in t} U\left(i, \mu\left(i\right)\right)$$

be the educational opportunity of an individual belonging to type $t \in T$. This definition of educational opportunity follows the ex-ante approach to EOp (Fleurbaey & Peragine, 2013; Roemer & Trannoy, 2015; Ramos & Van de Gaer, 2016), which evaluates individual opportunities in terms of expected outcomes conditional on individual characteristics.

Following Van de gaer (1993), part of the literature has converged on the idea that types should be ordered according to their observed opportunities. In other words, a type t is more advantaged that t' if $\bar{u}(t,\mu) > \bar{u}(t',\mu)$; we will call this the endogenous order. At the same time, after Atkinson & Bourguignon (1982), many authors have acknowledged the desirability of a type pre-order - like $\trianglelefteq$ defined above - that orders types in a way that accounts also for their needs, beyond the outcome of interest. Depending on the framework, one may favour one approach or the other. For example, to fight racial discrimination in the job market, it may not be sufficient to ensure equal expected outcome across ethnic groups. A more effective policy could be to grant and educational advantage to the discriminated groups: something that, as it will be cleared below, can be achieved using a pre-order. At the same time, the use of an endogenous order is in line with a broader idea of equality between groups which is relevant in many economic contexts. Here, we follow the approach based on the exogenous type order; the interested reader may refer to Appendix B for the alternative one.

Let us rearrange types according to the pre-order, so that $t_k \triangleleft t_{k+1}$ for all $k = \{1, ..., |T|-1\}$. The following axiom relies on $\trianglelefteq$, and a lexicographic order, to define a strong fairness requirement.

Axiom 2 (Strong Fairness)**.** There is no other assignment $\mu' \in M$ such that either $\bar{u}(t_1,\mu') > \bar{u}(t_1,\mu)$, or there exist $k \in \{2, ..., |T|\}$ such that, for all $j \in \{1, ..., k\}$, $\bar{u}(t_j,\mu') = \bar{u}(t_j,\mu)$ and $\bar{u}(t_k,\mu') > \bar{u}(t_k,\mu)$.

Axiom 2 expresses extreme aversion to inequality in types' expected match quality. In general, Axioms 1 and 2 are incompatible under stability. However, if the match quality measure is such that any two individuals attending the same school obtain the same educational outcome, then there always exist an assignment in $\mathcal{M}$ that satisfies the two axioms. We formalize this in the following lemma.[6]

Lemma 1. If match quality is such that, for all $i, i' \in I$ and $s \in S$, $U(i,s) = U(i',s)$, then there always exist an assignment $\mu \in \mathcal{M}$ that satisfies Axiom 1 and 2.

The formal proof is left to the reader, we provide here only a sketch. Intuitively, one can rename schools so that, for all $i \in I$, $U(i,s_1) \geq U(i,s_2) \geq ... \geq U\left(i, s_{|S|}\right)$, and types so that $t_1 \trianglelefteq t_2 \trianglelefteq ... \trianglelefteq t_{|T|}$. Then, it is sufficient to start with filling s_1's capacity with students from t_1, passing to s_2 if $q_{s_1} < |t_1|$, and to the following schools if necessary. Then we assign students from t_2 to the best schools with available seats and so on.

The reader may observe that in Lemma reflem:possibility there is no guarantee that the resulting assignment will also be stable. Given the incompatibility between the efficiency and fairness axioms, the central authority may start with listing all stable assignments, identifying the fair ones according to Axiom2, and choosing the most efficient among them. Although intuitive, this is not a trivial procedure because constructing the set of all stable assignments M can be extremely difficult for some school choice instances, as the number of stable assignments can be very big. It is well known in the

[6] The condition in Lemma 1 is not necessary, other match quality measures would ensure the result.

matching literature (see Gusfield & Irving, 1989) that finding all stable assignments, hence constructing M, is NP-complete (i.e. not solvable by a computer).

Despite the computational complexity, the problem of finding a stable assignment that satisfies Strong Fairness can be solved using Linear Programming (LP). In particular, Baïou & Balinski (2000) identifies a series of constraints one can use to check stability of a matching in an LP problem. Then it is sufficient to transform the lexicographic order of Axiom 2 into a linear function and set it as objective.

Formally, for any matching μ, let $x^\mu \in \{0,1\}^{|I|\times|S|}$ be such that $x^\mu_{(i,s)} = 1$ if $\mu(i) = s$ and $x^\mu_{(i,s)} = 0$. Let the match quality matrix U be such that all entries are strictly positive integers and denote $\bar{U} > \underline{U} \geq 1$ respectively the highest and the lowest match quality in U.[7] Abusing notation, let $\overline{u}(t_m,\mu) = \sum_{\substack{i\in t_m \\ s\in S}} \frac{1}{|t_m|} U(i,s) x^\mu_{(i,s)}$. The linear programming (LP) below, finds a stable assignment that satisfies Strong Fairness.

$$
\begin{array}{llll}
\max & \sum_{m=1}^{|T|} \overline{u}(t_m,\mu)\bar{U}^{|T|-m} & & (LP-1) \\
\text{s. t.} & \sum_{s\in S} x^\mu_{(i,s)} = 1 & \forall i \in I & \\
& \sum_{i\in I} x^\mu_{(i,s)} \leq q_s & \forall s \in S & \\
& \sum_{(i',s)\in \mathcal{C}(i,s)} x^\mu_{(i',s)} \geq q_s & \begin{array}{l}\forall s \in S, \\ \forall \mathcal{C}(i,s) \in \mathbb{C}_s\end{array} &
\end{array}
$$

Specifically, the first constraint imposes that any student is assigned to one school, the second checks that the assignment respects the quota of each school. The last condition of LP is checking if, for each student-school pair, (i,s) either student i is matched to a better school than s or the number of students who are ranked better than i in $\succ_s$ and admitted by the school s is equal to $|q_s|$. This condition, provided by Baïou & Balinski (2000), overlaps with the definition of stability; the reader may refer to Appendix C for more details on the notation. The correctness of constraints of LP above are proved by Baïou & Balinski (2000), the following proposition proves the correctness of the objective function.

Proposition 1. The linear programming LP-1 provides a stable matching that satisfies Strong Fairness.

Proof. See Appendix C.1. □

The same constraints can be used to formulate the linear programming that finds a stable allocation that satisfies Axiom 1.

Observe that LP-1 does not consider the total match quality. Consequently, one may end up sacrificing much of the aggregate educational opportunity for the sake of fairness. To limit this trade off, we can impose an additional constraint of the sort

[7] This restriction is necessary for a correct linear programming formulation. Observe also that if $\bar{U} = \underline{U}$ then any matching trivially satisfies Strong Fairness.

$$\sum_{i \in I} U(i,s) x_{(i,s)} \geq y$$

for some positive number y. Then, a grid search on the values of y can help the evaluator in choosing the right balance between the otherwise incompatible efficiency and fairness requirements.

One of the main drawbacks of using linear programming is that it can be difficult to understand and interpret the solution ex post. Linear programming is often considered a "black box" approach, as it can be challenging to understand how the solution was obtained and what specific constraints or factors led to the final outcome. In many real life contexts, we argue, the process of how an allocation is obtained is just as important as the outcome itself, and being able to understand and reconstruct all the steps that led to the outcome is crucial.

We conclude this section by considering weaker notions of efficiency and fairness. The properties below are interesting both as ways of weaken Axioms 1 and 2, and as guidelines for a central authority that wants to improve, if possible, an existing stable allocation. Before listing the axioms, let us recall a well known result in matching theory literature known as *Rural hospitals theorem* (Roth, 1986, 1984). This theorem states that if a school is not filling its capacity in some stable matching, then it will not fill its capacity in any other stable matching. In other words, under the assumption that $\mu \in M$, any $\mu' \in M$ can only be obtained via (consecutive) swaps of students within seats that are allocated in $\mu \in M$. With this result in mind, we can define the following notions of improvement, and the related axioms.

A matching $\mu \in M$ is a *net improvement* of $\mu' \in M$ if $U(k, \mu(k)) \geq U(k, \mu'(k))$ for all $k \in I \setminus \{i\}$, and $U(i, \mu(i)) > U(i, \mu'(i))$. In words, a net improvement is a series of swaps that increase match quality of a student, without hurting anyone else.

Axiom 3 (Efficiency)**.** There is no net improvement $\mu' \in M$.

In the special case where match quality corresponds to student's utility from a matching, Axiom 3 corresponds to the standard notion of Pareto efficiency. We continue with defining fairness improvements in terms of progressive Pigou-Dalton transfers.

A matching $\mu \in M$ is a *fairness improvement* of $\mu' \in M$ if $U(k, \mu(k)) \geq U(k, \mu'(k))$ for all $k \in I \setminus \{i, j\}$, with i and j such that $i \in t$, $j \in t'$, for some $t, t' \in T$ such that $t \lhd t'$, $U(i, \mu(i)) = U(i, \mu'(i)) + \delta$ and $U(j, \mu(j)) = U(j, \mu'(j)) - \delta$, for some $\delta > 0$.

The fairness improvement considers the type pre-order, so that improving the outcome of an individual in a worse off type, at the expenses of someone in a better off type, improves fairness of the matching. The corresponding local fairness condition can be expressed as follows.

Axiom 4 (Fairness)**.** There is no fairness improvement $\mu' \in M$.

The following section embeds Axioms 3 and 4 in a Social Welfare Function (SWFs), which is a tool that allow us to compare allocations. Interestingly, imposing additional standard properties on the SWF will allow us to define an algorithm that searches for the stable allocation that satisfies Axioms 3 and 4.

4 Normative approach

The previous section discussed the trade-off between Strong Efficiency and Strong Fairness and how the linear programming solution may be seen as unsatisfactory. In this section, a different approach is proposed. We begin by characterizing a family of Social Welfare Functions (SWFs), that is assumed to represent the preferences of a central authority, and is used to evaluate stable matchings. As before, we impose stability to be the minimal requirement for a desirable school allocation. Therefore, the central authority will aim at maximizing her SWF over the set of all stable assignments.

Denoting with M the set of stable matchings, $W : M \to \mathbb{R}$ is the function that measures the social welfare associated to a stable school assignment. In line with the preferences of an opportunity egalitarian social evaluator, we assume SWFs to satisfy the following axioms.

Monotonicity (MON) - For all $\mu, \mu' \in M$, if $U(i, \mu(i)) \geq U(i, \mu'(i))$ for all $i \in I$, with at least one strict inequality, then $W(\mu) \geq W(\mu')$.

Monotonicity is inspired by Efficiency. This property simply states that improving match quality of someone, without reducing it for anyone else, cannot worsen social welfare.

Additivity (ADD) - For all $\mu \in M$, there exist twice differentiable (almost everywhere) functions $\phi_i : \mathbb{R} \to \mathbb{R}$, for all $i \in I$, such that $W(\mu) \equiv \sum_{i \in I} \phi_i(U(i, \mu(i)))$.

The previous property imposes our social evaluation to be based on a standard utilitarian aggregation of match qualities. This axioms imposes separability across individuals, so that the comparison between two alternative allocations can be performed by comparing the changes in match quality of the sole individuals with a different matching. This allows for a more straightforward and transparent evaluation of the different allocations and makes it easier to understand how changes in match quality affect the overall matching.

Within type symmetry (SYM) - For all $\mu, \mu' \in M$, if there exists $t \in T$ and $i, i' \in t'$ such that $U(i', \mu'(i')) = U(i, \mu(i)) > U(i', \mu(i')) = U(i, \mu'(i))$, then $W(\mu) = W(\mu')$.

Within type symmetry implements the well-known anonymity principle, adapting it to a context in which circumstances do matter in the social evaluation. In words, this property requires social evaluation not to change if there is a permutation of match qualities involving individuals with the same characteristics. In other words, within a type, it does not matter who has a given match quality level.

Within types inequality neutrality (WTIN) - For all $\mu, \mu' \in M$, if there exist $t \in T$, $i, i' \in t$ and a positive real number δ such that $U(i, \mu'(i)) \geq U(i', \mu'(i'))$, $U(i, \mu(i)) = U(i, \mu'(i)) + \delta$, $U(i', \mu(i')) = U(i', \mu'(i')) - \delta$, and $U(i'', \mu(i'')) = U(i'', \mu'(i''))$ for all other $i'' \in I$, then $W(\mu) = W(\mu')$.

This property strengthen the idea that the match quality distribution within a type is matter of indifference for our social evaluation. Indeed, among individuals with the same circumstances, the final allocation should depend on their preferences, together with school priorities. Although one may argue that this is not the case for the latter, the former are expression of individual freedom which we intend to respect where possible.

Between types inequality aversion (BTIA) - For all $\mu, \mu' \in M$, if there exist two types $t, t' \in T$ such that $t \lhd t'$, $i \in t$, $i' \in t'$ and a positive real number δ such that $U(i', \mu(i')) = U(i', \mu'(i')) - \delta$, $U(i, \mu(i)) = U(i, \mu'(i)) + \delta$ and $U(i'', \mu(i'')) = U(i'', \mu'(i''))$ for all other $i'' \in I$, then $W(\mu) > W(\mu')$.

Between type inequality aversion mimics Axiom 4 in the previous section.

As also shown in Peragine (2004), the five axioms above characterize a family of linearly additive social welfare functions. Let ϕ' and ϕ'' denote respectively the first and second derivative of ϕ.

Lemma 2 (Peragine (2004))**.** For all $\mu \in M$, W satisfies MON, ADD, SYM, WTIN and BTIA if and only if

$$W(\mu) = \sum_{i \in I} \phi_i \left(U(i, \mu(i)) \right) \tag{1}$$

where the functions $\phi_i : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions: (i) for all $i \in I$, $\phi_i'' = 0$; (ii) for all $i, i' \in t$, $\phi_i' = \phi_{i'}' > 0$; and (iii) for all $i \in t$, $i' \in t'$ and $t, t' \in T$, if $t \lhd t'$, then $\phi_i' > \phi_{i'}' > 0$.

In the rest of this section, after introducing some necessary definitions and notation, we describe the Stable Opportunity Egalitarian (SOE) algorithm, which identifies the stable allocation that maximizes a given instance of eq. (1).

Irving *et al.* (1987) designed an algorithm to find a stable matching, in one-to-one matching setting, that maximizes preference satisfaction in both sides of the markets. Their algorithm relies on the concepts of "rotations" and "rotation poset" previously introduced by Irving & Leather (1986). Bansal *et al.* (2007) generalizes the concept of rotations to many-to-many matching markets. Cheng *et al.* (2008) show that, under particular restriction on what they call happiness measures, Irving *et al.* (1987) algorithm can be generalized to many-to-one matching setting. In our setting, we draw a parallel between Cheng *et al.* (2008) happiness function and our match quality measure, and use the proposed algorithm to maximize social welfare.

To properly define the algorithm, we need to introduce a series of technical concepts from the matching literature. We do so in the following subsection, which the expert reader may overlook.

4.1 Technical preliminaries

We begin with the well-known Deferred Acceptance algorithm (Gale & Shapley, 1962) in two different versions: one in which students propose to schools and schools choose who to admit, and one in which schools propose to students who then choose where to enrol.

We call *Student-proposing Deferred Acceptance* the following algorithm, and denote μ_I the resulting school allocation. (Step 1) Each student applies to her first ranked school. Each school collects the applications and temporarily assigns seats to applicants, one at a time, following the priority order up until the quota. Any remaining applicant is rejected. (Step k) Each student who was rejected in the previous step applies to her next most preferred school. Each school considers all the applicants - the new and the temporarily assigned ones from the previous step - and temporarily assigns its seats one at a time following the priority order up until the quota. Any remaining applicant is rejected. (Termination) If in the previous step no student was rejected, then the algorithm terminates and all the temporary assignments become final.

The same algorithm is called *School-proposing Deferred Acceptance* when schools and students are inverted in the process. We denote μ_S the resulting allocation.

Another necessary concept is preferences and priorities reduction, or pruning. This process identifies the student-school pairs that cannot be part of a stable assignment and removes them from the preferences and priority rankings. The first step of pruning is due to a well known result in matching theory literature, called *Rural hospitals theorem* (Roth, 1986, 1984). This theorem states that if a school is not filling its capacity in some stable matching, then it will not fill its capacity in any other stable matching. Moreover, this school will get always the same set of students in any stable matching. The second step of pruning is removing all the schools which are ranked above $\mu_I(i)$ in the preferences of student i and the schools which are ranked below $\mu_S(i)$. Since μ_I is the Pareto optimal allocation for students among all stable matchings, there is no stable matching μ where $\mu(i) P_i \mu_I(i)$. Conversely, for any student i, his assignment in any other stable matching $\mu(i)$ is weakly better than $\mu_S(i)$. Hence, for any student i, there is no stable matching μ where $\mu_S(i) P_i \mu(i)$. The third step of pruning is for the priority rankings of schools. The structure of the set of stable matchings holds symmetrically in terms of school's priority ranking, so that μ_S (resp. μ_I) is the best (resp. worst) stable matching from schools' perspective. The final step of pruning is to remove all non mutually acceptable student-school pairs on the priority rankings and the preferences.

Formally, the *pruning* of preferences and priority rankings in an instance $\mathcal{I}$ is obtained with the following procedure. (Step 1) For any school $s \in S$ such that $|\mu_I(s)| < q_s$: (i) remove from $\succ_s$ all students $i \notin \mu_I(s)$; (ii) for all $i \in \mu_I(s)$ remove from P_i all schools $s' \neq s$. (Step 2) For any student $i \in I$, remove from P_i all schools $s \in S$ such that $s P_i \mu_I(i)$ or $\mu_S(i) P_i s$. (Step 3) For any school $s \in S$, remove from $\succ_s$ all students $i \in I$ such that $i \succ_s i'$ for all $i' \in \mu_S(s)$, or $i'' \succ_s i$, for all $i'' \in \mu_I(s)$. (Step 4) For each $s \in S$ and $i \in I$: (i) if $i \in \succ_s$ but $s \notin P_i$, remove i from $\succ_s$; (ii) if $s \in P_i$ but $i \notin \succ_s$, remove s from P_i.

The pair of pruned preferences and priorities, denoted $(P^\star, \succ^\star)$ define the set of admissible student-school pairs in a stable matching.

We call *graph* of a matching μ, the directed graph $G_\mu = (V_\mu, E_\mu)$ in which: (i) the set of vertices, V_μ, is formed by the set of student-school pairs, $(i, s) \in I \times S$, such that

i is the worst student for school s according to $\succ_s^\star$ and $\mu(i) = s$, and (ii) E_μ is a set of oriented edges such that: there is an edge from a vertex (i, s) to a vertex (i', s') whenever s' is the second best school after $\mu(i)$, according to $P_i^\star$. Endowed with the notion of graph of a matching, we can define the following key ingredient of our algorithm.

Let μ denote a school assignment. A *rotation*

$$r = \langle (i_1, s_1), (i_2, s_2), ..., (i_n, s_n) \rangle$$

on $G_\mu = (V_\mu, E_\mu)$, or *exposed rotation* in μ, is a sequence of vertices in V_μ such that for all $j \in \{1, ..., n\}$, there is an oriented edge from (i_j, s_j) to (i_{j+1}, s_{j+1}) where j is taken modulo n. [8] We also call r the exposed rotation in μ starting from i_1.

In words, a rotation on a graph is a cycle that, starting from a given vertex, follows a sequence of edges until it reaches the starting point.

Let $V_\mu \setminus r$ denote the set of student-school pairs which are not in r. A matching μ' *eliminates* an exposed rotation r in μ - denoted $\mu' = \mu \setminus r$ - if, for all student-school pairs in $r = \langle (i_1, s_1), (i_2, s_2), ..., (i_n, s_n) \rangle$, $\mu'(i_j) = s_{j+1}$ where j is taken modulo n, and for all student-school pairs in $V_\mu \setminus r$, $\mu(i) = \mu'(i)$. In words, a rotation defines a sequence of re-allocations in which student j (who was the least preferred student by the school she is assigned to) goes to his second most preferred school, taking the spot of $j + 1$ (who was the least preferred student by the school he is assigned to) who goes to his second most preferred school and takes the spot of $j + 2$ who then goes to his second most preferred and so on. If we implement this sequence of transfers, we obtain a new assignment which is said to be the matching that eliminates that rotation. Eliminating an exposed rotation r in μ creates a new assignment $\mu' = \mu \setminus r$. Starting from a student i such that $\mu'(i) \neq \mu_S(i)$, we can expose new rotations in μ'. This process of eliminating an exposed rotation, and exposing new rotations, allows us to define the set of all exposed rotations, starting from a given matching μ. Since some rotations will be exposed only after eliminating others, it is useful to talk about *successors* and *predecessors* of a rotation. We say that a rotation ρ is a successor of r - denoted $r < \rho$ - if ρ is an exposed rotation in $\mu \setminus r$ and it is not possible to expose ρ without eliminating r. Observe that, since there may be multiple exposed rotations in an assignment, a rotation ρ may be only exposed after eliminating multiple rotations.

For each rotation $\rho = \langle (i_1, s_1), (i_2, s_2), ..., (i_n, s_n) \rangle$ exposed on μ, we define a *weight* $\omega(\rho) = W(\mu \setminus \rho) - W(\mu)$. Observe that, because match quality satisfies IND and W is linearly additive, $\omega(\rho) = \sum_{i_j \in \rho} \phi_{i_j} \left(U(i_j, s_{j+1}) \right) - \phi_{i_j} \left(U(i_j, s_j) \right)$ for j taken modulo n.

Given an instance $\mathcal{I} = \langle I, S, P, \succ, q, T, \lhd, U \rangle$, denote $\Pi(\mathcal{I}, \mu)$ the set of all exposed rotations starting from μ. Notice that the relation $<$ defines a partial order on $\Pi(\mathcal{I}, \mu)$, so that $(\Pi(\mathcal{I}, \mu), <)$ forms a partially ordered set. A closed set in the poset $\Pi(\mathcal{I}, \mu)$ is a subset $C(\mathcal{I}, \mu)$ of $\Pi(\mathcal{I}, \mu)$ such that

$$\rho \in C(\mathcal{I}, \mu), r < \rho \Rightarrow r \in C(\mathcal{I}, \mu)$$

[8] That is, if $j = n$, then $j + 1 = 1$.

In words, a subset of rotations $C(\mathcal{I},\mu) \subset \Pi(\mathcal{I},\mu)$ is closed if it contains all the predecessors of its elements. Following Bansal *et al.* (2007), we can formalize the following result.

Lemma 3 (Bansal *et al.* (2007))**.** Let $\mathcal{I}$ be an instance of school choice problem with EOp components and μ_I its student-proposing DA assignment. There is a one-to-one correspondence between the closed subsets of $\Pi(\mathcal{I},\mu_I)$ and the set of all stable matchings of $\mathcal{I}$: each closed subset $C(\mathcal{I},\mu_I)$ of $\Pi(\mathcal{I},\mu_I)$ corresponds to a unique stable matching generated by eliminating all the rotations in $C(\mathcal{I},\mu_I)$.

The previous lemma provides a powerful result which tells us that we can somehow explore the set of all stable matchings via looking at all exposed rotations, starting from the student-proposing DA assignment. While listing all the stable matchings of $\mathcal{I}$ is NP-hard (Gusfield & Irving, 1989), $\Pi(\mathcal{I},\mu_I)$ can be constructed with an efficient algorithm[9] (see Cheng *et al.*, 2008) in the following way.

Step 0: Run student proposing and school proposing DAs, find the respective matchings μ_I and μ_S, and prune preferences and priority rankings.

Step 1: (1.1) Form the graph of matching μ_I - G_{μ_I}. Starting from a student i who is not matched to $\mu_S(i)$, find an exposed rotation ρ_1. (1.2) Add this rotation to the rotation poset $(\Pi(\mathcal{I},\mu_I))$ and compute $\omega(\rho_1) = W(\mu_I \setminus \rho_1) - W(\mu_I)$.

Step 2: (2.1) Form the graph of $\mu_I \setminus \rho_1$ and, starting from a student i who is not matched to $\mu_S(i)$ find an exposed rotation ρ_2. (2.2) Add this rotation to the rotation poset and compute its weight $\omega(\rho_2) = W(\mu_I \setminus \{\rho_1,\rho_2\}) - W(\mu_I \setminus \rho_1)$.

Step k: (k.1) Form the graph of $\mu_I \setminus \{\rho_1, ...\rho_{k-1}\}$ and expose a rotation ρ_k. (k.2) Add this rotation in the rotation poset and compute its weight.

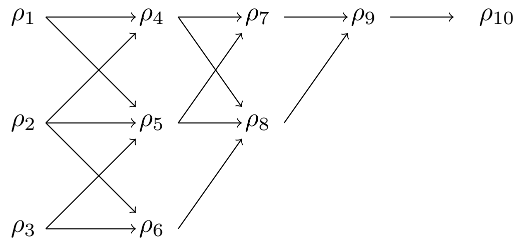
Termination: Terminate this procedure until, after eliminating a rotation, you obtain μ_S.

No student-school pair can belong to more than one rotation (Cheng *et al.*, 2008). Consequently, there are at most $|I| \cdot |S|$ rotations. After finding all the rotations, we need to choose a closed subset of $\Pi(\mathcal{I},\mu_I)$ such that eliminating all the rotations in this subset will maximize our social evaluation function. A graphical example of a rotation poset is provided in Figure 1. The graph in is composed of vertices, which correspond to the rotations, and edges that connect each rotation to its successors. For a more detailed explanation on how to construct the such a graph, see the example in Appendix D.

Finding the maximum closed subset of $\Pi(\mathcal{I},\mu_I)$ is a selection problem like the one introduced by Balinski (1970) and Rhys (1970). Picard (1976) shows that this problem can be easily solved using Linear Programming. In particular, let n be the number of rotations in $\Pi(\mathcal{I},\mu_I)$, then finding the maximum closed subset is equivalent to finding the vector $\mathbf{x} \in \{0,1\}^n$ that solves

[9] An example of this algorithm is reported in the Appendix.

Figure 1: A graphical example of a rotation poset.



$$\max_{\mathbf{x}} z = \sum_{j=1}^{n} \omega\left(\rho_j\right) x_j + \lambda \sum_{j=1}^{n} \sum_{h=1}^{n} a_{jh} x_j \left(-1 + x_h\right) \tag{2}$$

where $a_{jh} = 1$ if, on the rotation graph, there is a directed edge from ρ_h to ρ_j (i. e. ρ_j is a successor of ρ_h) and $a_{jh} = 0$ otherwise, and λ must be an arbitrarily large real number.[10] In words, $x_j = 1$ means that the rotation j is in the subset. For each possible subset of $\Pi(\mathcal{I}, \mu_I)$, the first element of Eq. 2 sums the weights of the rotations in the subset and the second element checks that, for each of these rotations, the respective predecessors are included in the subset. Suppose that ρ_h is a predecessor of ρ_j, ρ_j is in the subset but ρ_h is not, then $a_{jh} x_j \left(-1 + x_h\right) = 1 \cdot 1(-1 + 0) = -1$.

Picard (1976) shows that there is a more efficient way of solving this problem by reducing the graph of the rotation poset to a network flow graph and applying a min-cut algorithm. This is the procedure we suggest in our algorithm. However, to maintain a simple exposition, we do not include the details of this methodology in the main text, the interested reader may refer to Appendix E.

4.2 The algorithm

We introduce now the Stable Opportunity Egalitarian (SOE) algorithm, which is structured as follows. First, set a functional form for W. Second, construct the rotation poset. Third, construct network flow of rotations. Fourth, apply an efficient minimum cut algorithm to identify the closed subset of rotation we need to eliminate to maximise W. Fifth, return the optimal allocation. The pseudo-code of the SOE algorithm is as follows.

Let us recall that there are at most $|I| \cdot |S|$ rotations. Let t represent the longest running time that takes to calculate $\omega(\rho)$ for some rotation ρ. It takes at most $|I|^4$ running time to find the optimal closed subset of $\Pi\left(\mathcal{I}, \mu_I\right)$ (see Gusfield & Irving (1989), page 129-133). Putting all these calculations together, we can conclude that the SOE algorithm takes at most $|I||S|t + |I|^4$ running time; this is an efficient running time performance.

The following two statements prove the correctness of the SOE algorithm.

[10] It must be $\lambda > \max\{\omega(\rho_j)\}_{j=1}^{n}$ to ensure that the optimal solution satisfies $\sum_{j=1}^{n} \sum_{h=1}^{n} a_{jh} x_j \left(-1 + x_h\right) = 0$, which means that we have found a closed subset.

Algorithm 1 The SOE algorithm (Cheng *et al.*, 2008)

Input: W and $\mathcal{I} = \langle I, S, P, \succ, q, T, \trianglelefteq, U\rangle$
Compute μ_I and μ_S.
Prune preferences of students and priority rankings of schools;
$\mu \leftarrow \mu_I$, $\Pi(\mathcal{I}, \mu) = \emptyset$
while $\mu \neq \mu_S$ **do**
    Find $i^* \in I$ such that $\mu(i^*) \neq \mu_S(i^*)$
    Find a rotation ρ exposed in μ starting from i^*
    Add ρ to $\Pi(\mathcal{I}, \mu)$
    $\omega(\rho) \leftarrow W(\mu \setminus \rho) - W(\mu)$
    $\mu \leftarrow \mu \setminus \rho$
end while
Construct the network flow of rotations and use min-cut to find $C^* = \arg\max_{C \subset \Pi(\mathcal{I},\mu)} \sum_{r \in C} \omega(r)$;
$\mu \leftarrow \mu_I \setminus C^*$
Return μ

Lemma 4. Let $\mathcal{I}$ be a school choice instance. Let C be a closed subset of $\Pi(\mathcal{I}, \mu_I)$, and μ be the stable matching obtained by eliminating all the rotations in C. If U satisfies IND, and W satisfies MON, ADD, WTIN, SYM and BTIA, then $W(\mu) = W(\mu_I) + \sum_{r \in C} \omega(r)$.

The previous result is a direct implication of the additive decomposability of Eq. 1, combined with the independence property satisfied by the match quality measures in our setting. We now provide the main result of the paper.

Theorem 1. Let $\mathcal{I} = \langle I, S, P, \succ, q, T, \lhd, U\rangle$ be an instance of centralized school choice setting with EOp components. Let $W : M \to \mathbb{R}$ be a social welfare function that satisfies MON, ADD, WTIN, SYM and BTIA. Then, the Stable Opportunity Egalitarian algorithm maximizes W over the set of stable assignments for $\mathcal{I}$.

Proof. By Lemma 4, finding a closed subset C that maximizes W is equivalent to finding a matching that maximizes our social evaluation function. By Lemma 3, this matching will be stable. $\square$

5 Discussion

In this section we comment on relevant features of the SOE algorithm, and its link with the normative principles behind the EOp paradigm.

Let us begin by recalling that Student-proposing DA delivers the best allocation, among stable ones, in terms of preference satisfaction. Consequently, if match quality corresponds to student's preference satisfaction,[11] then μ_I is optimal, for any SWF as in eq. (1). The reader may notice that with this type of match quality, it is as if the central planner aligns his preferences to those of the students. In the EOp paradigm, this situation corresponds to a strong reward principle which calls for preservation of all

[11] That is, for all $i, i' \in I$ and $s, s' \in S$, sP_is' implies $U(i,s) > U(i,s')$, and $P_i(s) = P_{i'}(s')$ implies $U(i,s) = U(i',s')$.

those inequalities due to factors within individual control: preferences in our context. To put it differently, the use of Student-proposing DA to allocate students can be justified by the normative principle of respecting students' preferences and holding individuals responsible for them (Dworkin, 1981b).

The SOE algorithm deviates from either μ_I or μ_S whenever the central authority's preferences do not coincide with those of the students or schools. This is the case, for example, if the evaluator does not want to hold individuals responsible for their preferences, which can be influenced by factors out of individual control. In the matching literature, affirmative action policies are directly or indirectly aimed at modifying priority rankings in favour of particular groups. In our context, we achieve a similar result by shifting the attention of the social evaluator toward other preferences of which the match quality measure, in combination with the social welfare function, offer a representation. Equalizing opportunity is tightly linked with the rationale behind affirmative action policies. Intuitively, if we want to improve chances for disadvantaged students to get into preferred schools, it may be sufficient to modify the schools' priority rankings by upward moving these students. Observe that we can still focus on multiple groups at the same time by calibrating the *bonuses* given to students from different types. In line with the principle of respect for students' preferences, we can run Student-proposing DA and implement the resulting allocation, call it $\mu*$. Despite the adverse effect this can have in theory (see Kojima, 2012), we should expect this to improve the situation for the disadvantaged groups. The matching $\mu*$ may fail to satisfy the Stability requirement which is defined in terms of the original preferences and priority rankings. However, if $\mu*$ satisfies Stability, then it is possible to define a match quality measure U and a social welfare function W such that the SOE algorithm identifies $\mu*$ as the optimal allocation.

The SOE algorithm has the advantage of being a deterministic procedure in which we can reconstruct all the passages that lead to a given allocation. Intuitively, the starting point of the algorithm is always the best possible allocation for students. Then, it proceeds with sacrificing student's preference satisfaction in order to maximize the evaluator's preferences, represented by the social welfare function. In this sense, when moving away from Student-proposing DA, the algorithm trades off Pareto efficiency and maximization of the evaluator's preferences. It is worth underlining here that, in line with our discussion in Section 3, we can use linear programming to maximize W under the stability constraints. Abstracting here form all issues related to the computational complexity of the two procedures, the SOE algorithm is clearly more transparent and easier to back-track. For sensitive matters like school allocation, we believe this constitutes a strong motivation for preferring our deterministic procedure to a linear programming solution.

We conclude this section by discussing other applications of the SOE algorithm. Problems like doctor-hospital or refugees allocation are other instances of many-to-one matching problems in which opportunity egalitarian principles find application. In our setting we focus on the problem of equalizing student's opportunity for good education. Social planners may be concerned with the symmetric problem of equalizing hospital's opportunity for good doctors, with the aim of reducing regional disparities in the health care system. The SOE algorithm offers a solution to this problem as well. Clearly, the same holds for any many-to-one matching setting in which can be expressed, as we do in this paper, as a problem of maximization of a linear social welfare function over the set of stable matchings.

6 Conclusion

We considered a relevant case of many-to-one matching market - the standard school choice setting - augmented with a partition of the population into types and a measure of match quality. We introduced discussed three desirable properties for a school assignment: stability, efficiency and fairness. The former corresponds to the standard requirement in matching theory and the second property has already been proposed by Abdulkadiroglu *et al.* (2021). Our fairness requirement, inspired by the opportunity egalitarian paradigm, is novel and based on the idea of minimizing the inequality in types' expected educational outcome. We highlighted the incompatibility between these three requirements and provided two approaches to solve it. The first solution is a linear programming to identify the fair, or the efficient, allocation within the set of stable ones. The second solution, which is also our favourite, defines a family of opportunity egalitarian social welfare functions, together with an algorithm that maximizes social welfare over the set of stable matchings.

To the best of our knowledge, this is the first paper to draw a clear connection between the school choice and the equality of opportunity literature, and we believe this to be only a first step toward a more consistent dialogue between those two literatures. There is great scope for further exploring how existing algorithms in the matching literature can be used to solve complex fairness issues, like equality of opportunity, that go beyond the standard aversion to inequality. The literature has proposed different families of opportunity egalitarian social welfare functions, many of which fail to satisfy the independence property. Identifying algorithmic solutions to the problem of maximizing non-linear opportunity egalitarian social welfare functions over the set of stable matchings is highly ranked in our research agenda.

Finally, we underline that problems like doctor-hospital or refugees allocation are other instances of many-to-one matching problems in which opportunity egalitarian principles find application. Clearly, as far as one can express the problem as a maximization of a linear social welfare function over the set of stable matchings the SOE algorithm can be used to obtain a stable welfare maximizing allocation. This greatly enlarges the possible application of our approach, opening the way to further investigations on algorithmic solutions that satisfy complex fairness principles like the opportunity egalitarian one.

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Appendix A Affirmative action policies versus equality of opportunity

We propose here an simple example of how affirmative action policies may not align with opportunity egalitarianism. Consider a simple scenario with six students and three schools, each offering two seats. The schools differ in quality, which we approximate by using the average probability of a student receiving an offer from a top college. School 1 has probability of 4%, school 2 has 10%, and school 3 has 20%. Suppose that students 1 and 2 belong to the most disadvantaged population subgroup, students 5 and 6 belong to the most advantaged subgroup, while students 3 and 4 belong to a population subgroup within these two extremes. Let preferences of students (i) and priority rankings of schools (s) be:

$$\begin{array}{lcl} i_1 = i_2 & : & s_3, s_1, s_2 \\ i_3 = i_4 = i_5 = i_6 & : & s_3, s_2, s_1 \\ s_1 & : & i_6, i_5, i_1, i_2, i_3, i_4 \\ s_2 = s_3 & : & i_6, i_5, i_4, i_3, i_2, i_1 \end{array}$$

where, for example, s_3 is the most preferred school for i_1 and i_2, and i_4 is the least preferred student for s_1. Following a standard approach in the literature, we represent the opportunity distribution with a vector $\mathbf{p} = (p_1, p_2, p_3)$, where p_t is the expected probability for a random student of group t of receiving an offer from a top college. Let us consider the opportunity distributions generated by the following three allocations:

$$\begin{bmatrix} & DA & \\ s_1 & \leftarrow & i_1, i_2 \\ s_2 & \leftarrow & i_3, i_4 \\ s_3 & \leftarrow & i_5, i_6 \end{bmatrix} \begin{bmatrix} & EOp & \\ s_1 & \leftarrow & i_1, i_2 \\ s_2 & \leftarrow & i_3, i_4 \\ s_3 & \leftarrow & i_5, i_6 \end{bmatrix} \begin{bmatrix} & AA & \\ s_1 & \leftarrow & i_1, i_2 \\ s_2 & \leftarrow & i_3, i_4 \\ s_3 & \leftarrow & i_5, i_6 \end{bmatrix}$$

where DA is the standard Gale & Shapley (1962)'s Deferred Acceptance (DA) algorithm explained in Section 4, EOp is obtained by exchanging the school assignment of students 2 and 6, and AA is obtained when school 3 reserves one seat to students from the most disadvantaged group (a minority quota) and DA is implemented. Observe that $\mathbf{p}^{DA} = (4, 10, 20)$, $\mathbf{p}^{EOp} = (12, 10, 12)$ and $\mathbf{p}^{AA} = (12, 7, 15)$. The Gini coefficient of the opportunity distribution is 0.31 for $\mathbf{p}^{DA}$, 0.16 for $\mathbf{p}^{AA}$ and 0.04 for $\mathbf{p}^{EOp}$. The comparison in terms of equality of opportunity is neat, revealing that Deferred Acceptance does not guarantee distributional fairness. It is interesting to also notice how, because of the focus on a single group, the affirmative action policy, in this example, puts the burden of fairness also on the second group who is not as advantaged as the third one. This consequence may be normatively unappealing.[12] By giving importance to all groups, the opportunity egalitarian approach we implement in this paper tries to also limit this last issue.

[12] See also Students for Fair Admissions, Inc. v. President and Fellows of Harvard College (Docket 20–1199), for a recent real life example.

Appendix B Minimizing between group inequality

In this appendix, we discuss an alternative fairness axiom. Following Van de gaer (1993), part of the literature has converged on the idea that types should be ordered according to their observed opportunities. In other words, a type t is more advantaged that t' if $\bar{u}(t,\mu) > \bar{u}(t',\mu)$, we will call this the endogenous order. The use of an endogenous order is in line with a broader idea of equality between groups which is relevant in many economic contexts, hence worth of our interest.

The following axiom formalizes the fairness requirement based on the endogenous type order.

Axiom 5 (Global group egalitarian)**.** There is no other assignment $\mu' \in M$ such that $\sum_{j,k=1}^{|T|} |\bar{u}(t_j,\mu') - \bar{u}(t_k,\mu')|$ is strictly lower.

While it is clear that Axiom 1 is not compatible with Axiom 5, the following lemma identifies sufficient conditions for the existence of an assignment that satisfies them both.

Lemma 5. Let $\mathcal{I}$ be such that: *(i)* $|t| = |t'|$ for any $t, t' \in T$, and *(ii) there exists* $z_s \in \mathbb{N}_{++}$ *such that* $q_s = z_s|T|$ *for all* $s \in S$. If $U : I \times S \to \mathbb{R}$ is such that, for all $i, i' \in I$ and $s \in S$, $U(i,s) = U(i',s)$, then there always exist an assignment $\mu \in \mathcal{M}$ that satisfies Axiom 1 and 5.

Intuitively, condition (ii) in Lemma 5 ensures that each school can guarantee an equal representation of types. Therefore, if one fills up schools, starting with the one of highest match quality, and ensuring equal representation, the final allocation will satisfy both Axiom 1 and 5. Once again, though, there is no guarantee that the resulting assignment will also be stable.

Finding a stable allocation that satisfies Axiom 5 can also be written as a Linear Programming. From the one in Section 3, it is sufficient to substitute

$$\min \sum_{j,k=1}^{|T|} \left| \sum_{i\in t_j, s\in S} \frac{1}{|t_j|} U(i,s)\, x_{(i,s)} - \sum_{i\in t_k, s\in S} \frac{1}{|t_k|} U(i,s)\, x_{(i,s)} \right|$$

as objective function.

Appendix C Linear programming for a fair stable allocation

In this appendix, we provide additional details on the linear programming formulation to find a stable allocation satisfying Strong Fairness. This formulation is provided by Baïou & Balinski (2000).

A school choice graph $G = (V,E)$ is a directed graph. The set of vertices is composed of mutually acceptable school-student pairs; in our case $V = I \times S$. There is an edge from $v = (i,s)$ to $v' = (i',s')$, $e = (v,v') \in E$, if either $i = i'$ and $s' P_i s$ or if $s = s'$

and $i' \succ_s i$. The following example represents a school choice graph of an instance: $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, with preferences and priority rankings

P_{i_1}	P_{i_2}	P_{i_3}	s_1	s_2	s_3
s_3	s_1	s_1	i_3	i_1	i_1
s_2	s_3	s_2	i_2	i_3	i_2
s_1	s_2	s_3	i_1	i_2	i_3

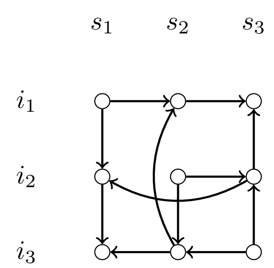


Figure 2: The graph $G = (V, E)$ of a school choice problem

On the graph above, the arcs implied by transitivity are omitted. For example, on the graph, it is obvious that $s_3 P_{i_1} s_1$.

A vertex v' is a successor of another vertex v if there is an edge from v to v', i.e. $e = (v, v') \in E$. For example on the graph above, (s_2, i_1) and (s_1, i_2) are the successors of (s_1, i_1). Let V^+ be the set of all vertices $v = (s, i) \in V$ that have at least $q_s - 1$ successors such that for each successor $v' = (s', i')$ of $v = (s, i)$, $s = s'$. V^- is the set of all other vertices, i.e. $V^- = V \setminus V^+$.

For each school $s \in S$, a *shaft* of $s \in S$ consists of a vertex $(i, s) \in V^+$ which is the base, and all of its successors in V^+. For each school, there may exist multiple student-school pairs in V^+ but only one shaft, which we denote $\mathcal{S}(i, s)$ to highlight that $(i, s) \in V^+$ is the base.

A *tooth* $\mathcal{T}$ of $(i, s) \in V$ - written $\mathcal{T}(i, s)$ - consists of the vertex (i, s) which is the source, and all of its successors (i', s') such that $i = i'$.

A *comb* $\mathcal{C}$ of $s \in S$ is the union of its shaft $\mathcal{S}(i, s)$ and exactly q_s teeth of $(i', s) \in \mathcal{S}(i, s)$, including $\mathcal{T}(i, s)$. It is written as $\mathcal{C}(i, s)$. The teeth are decided by the priority ranking of s. Hence if $\mathcal{T}(i'', s) \in \mathcal{C}(i, s)$, then any teeth $\mathcal{T}(i', s)$ such that $i' \succ_s i''$ are also in $\mathcal{C}(i, s)$. The set of all combs of s is $\mathbb{C}_s$.

The fourth condition of LP in Section 3:

$$\sum_{(i',s) \in \mathcal{C}(i,s)} x_{(i',s)} \geq q_s \qquad \forall s \in S, \forall \mathcal{C}(i, s) \in \mathbb{C}_s$$

checks that each comb of s contains at least q_s (s, i) vertices such that $\mu(i) = s$ or $x_{(i,s)} = 1$. As shown in Theorem 3 of Baïou & Balinski (2000), this condition is necessary and sufficient for the assignment to be stable.

C.1 Proof of Proposition 1

Assume that $\mu \in M$ satisfies Strong Fairness. By Baïou & Balinski (2000), the incidence vector of μ, x^{μ}, satisfies three constraints of LP-1. We need to show that

$$\sum_{m=1}^{|T|} \overline{u}(t_m,\mu)\bar{U}^{|T|-m} \geq \sum_{m=1}^{|T|} \overline{u}(t_m,\mu')\bar{U}^{|T|-m} \qquad \forall \mu' \in M \tag{3}$$

We proceed by induction. Let us first prove that if $\overline{u}(t_1,\mu) > \overline{u}(t_1,\mu')$, then (3) holds.

Observe that the maximum value of $\overline{u}(t_m,\mu')$ is $\bar{U}$ for any $m \in \{2,\dots,|T|\}$. Hence,

$$\bar{U}\left(\frac{\bar{U}^{|T|-1}-1}{\bar{U}-1}\right) = \sup \sum_{m=2}^{|T|} \overline{u}(t_m,\mu')\bar{U}^{|T|-m}$$

Let $\underline{U}$ be the minimum value on U, then

$$\underline{U}\left(\frac{\bar{U}^{|T|-1}-1}{\bar{U}-1}\right) = \inf \sum_{m=2}^{|T|} \overline{u}(t_m,\mu)\bar{U}^{|T|-m}$$

Suppose, by contradiction, that (3) is false so that

$$\overline{u}(t_1,\mu)\bar{U}^{|T|-1} + \underline{U}\left(\frac{\bar{U}^{|T|-1}-1}{\bar{U}-1}\right) < \overline{u}(t_1,\mu')\bar{U}^{|T|-1} + \bar{U}\left(\frac{\bar{U}^{|T|-1}-1}{\bar{U}-1}\right)$$

.

We then have,

$$\left(\overline{u}(t_1,\mu) - \overline{u}(t_1,\mu')\right)\left[\bar{U}^{|T|} - \bar{U}^{|T|-1}\right] < (\bar{U} - \underline{U})\left[\bar{U}^{|T|-1} - 1\right]$$

which, since $\overline{u}(t_1,\mu) - \overline{u}(t_1,\mu') \geq 1$, leads us to

$$\bar{U}^{|T|} - \bar{U}^{|T|-1} < (\bar{U} - \underline{U})\left[\bar{U}^{|T|-1} - 1\right]$$

$$\bar{U}^{|T|-1}\left(\underline{U} - 1\right) < \underline{U} - \bar{U}$$

$$\bar{U}^{|T|-1}\left(\underline{U} - 1\right) < 0$$

A contradiction, since $1 \leq \underline{U} \leq \bar{U}$.

Now, we assume that the statement is true for the case $n = k$. We want to show that it holds for $n = k+1$.

Let $\overline{u}(t_n,\mu) = \overline{u}(t_n,\mu')$ for $n = 1,..,k$ and assume $\overline{u}(t_{k+1},\mu) > \overline{u}(t_{k+1},\mu')$. We need to show that (3) holds. Observe that,

$$\bar{U}\left(\frac{\bar{U}^{|T|-k-1}-1}{\bar{U}-1}\right) = \sup \sum_{m=k+2}^{|T|} \overline{u}(t_m,\mu')\bar{U}^{|T|-m}$$

and

$$\underline{U}\left(\frac{\bar{U}^{|T|-k-1}-1}{\bar{U}-1}\right)=\inf\sum_{m=k+2}^{|T|}\overline{u}(t_m,\mu)\bar{U}^{|T|-m}$$

By contradiction, assume that (3) is false, so that

$$\overline{u}(t_{k+1},\mu)\bar{U}^{|T|-k-1}+\underline{U}\left(\frac{\bar{U}^{|T|-k-1}-1}{\bar{U}-1}\right)<\overline{u}(t_{k+1},\mu')\bar{U}^{|T|-k-1}+\bar{U}\left(\frac{\bar{U}^{|T|-k-1}-1}{\bar{U}-1}\right)$$

the same algebraic operations lead to the desired contradiction.

Assume now that (3) holds, we want to show that $\mu\in M$ satisfies Strong Fairness. By contradiction, assume that there exist $\mu'\in M$ and $1\leq k\leq|T|$ such that $\overline{u}(t_j,\mu)=\overline{u}(t_j,\mu')$ for all $j<k$ and $\overline{u}(t_k,\mu)<\overline{u}(t_k,\mu')$. We can rewrite (3) as

$$\begin{aligned}\sum_{m=1}^{k-1}\overline{u}(t_m,\mu)\bar{U}^{|T|-m}+\overline{u}(t_k,\mu)\bar{U}^{|T|-k}+\sum_{m=k+1}^{|T|}\overline{u}(t_m,\mu)\bar{U}^{|T|-m}\geq\\ \sum_{m=1}^{k-1}\overline{u}(t_m,\mu')\bar{U}^{|T|-m}+\overline{u}(t_k,\mu')\bar{U}^{|T|-k}+\sum_{m=k+1}^{|T|}\overline{u}(t_m,\mu')\bar{U}^{|T|-m}\end{aligned}\tag{4}$$

which implies

$$\left(\overline{u}(t_k,\mu')-\overline{u}(t_k,\mu)\right)\bar{U}^{|T|-k}\leq\sum_{m=k+1}^{|T|}\left(\overline{u}(t_m,\mu)-\overline{u}(t_m,\mu')\right)\bar{U}^{|T|-m}$$

the easiest way for the above inequality to hold is when $\overline{u}(t_k,\mu')-\overline{u}(t_k,\mu)=1$ and all differences in the right hand side correspond to $\bar{U}-\underline{U}$. Hence, it must be

$$\bar{U}^{|T|-k}\leq\left(\bar{U}-\underline{U}\right)\left(\frac{\bar{U}^{|T|-k}-1}{\bar{U}-1}\right)$$

which leads to

$$\left(\underline{U}-1\right)\bar{U}^{|T|-k}\leq\underline{U}-\bar{U}$$

the desired contradiction.

Appendix D Example of rotation poset

In this appendix we adapt an example from Irving *et al.* (1987) to our school choice setting.

Let there be 8 students and 8 schools, each with capacity of one. Let the preferences and priority rankings be as follows.

$$
\begin{array}{llll}
i_1: & s_3, s_1, s_5, s_7, s_4, s_2, s_8, s_6 & s_1: & i_4, i_3, i_8, i_1, i_2, i_5, i_7, i_6 \\
i_2: & s_6, s_1, s_3, s_4, s_8, s_7, s_5, s_2 & s_2: & i_3, i_7, i_5, i_8, i_6, i_4, i_1, i_2 \\
i_3: & s_7, s_4, s_3, s_6, s_5, s_1, s_2, s_8 & s_3: & i_7, i_5, i_8, i_3, i_6, i_2, i_1, i_4 \\
i_4: & s_5, s_3, s_8, s_2, s_6, s_1, s_4, s_7 & s_4: & i_6, i_4, i_2, i_7, i_3, i_1, i_5, i_8 \\
i_5: & s_4, s_1, s_2, s_8, s_7, s_3, s_6, s_5 & s_5: & i_8, i_7, i_1, i_5, i_6, i_4, i_3, i_2 \\
i_6: & s_6, s_2, s_5, s_7, s_8, s_4, s_3, s_1 & s_6: & i_5, i_4, i_7, i_6, i_2, i_8, i_3, i_1 \\
i_7: & s_7, s_8, s_1, s_6, s_2, s_3, s_4, s_5 & s_7: & i_1, i_4, i_5, i_6, i_2, i_8, i_3, i_7 \\
i_8: & s_2, s_6, s_7, s_1, s_8, s_3, s_4, s_5 & s_8: & i_2, i_5, i_4, i_3, i_7, i_8, i_1, i_6
\end{array}
$$

Student proposing and school proposing DA give:

$$
\mu_I : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_2 & i_8 & i_1 & i_5 & i_4 & i_6 & i_3 & i_7 \end{array}
$$

$$
\mu_S : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_4 & i_3 & i_7 & i_6 & i_8 & i_5 & i_1 & i_2 \end{array}
$$

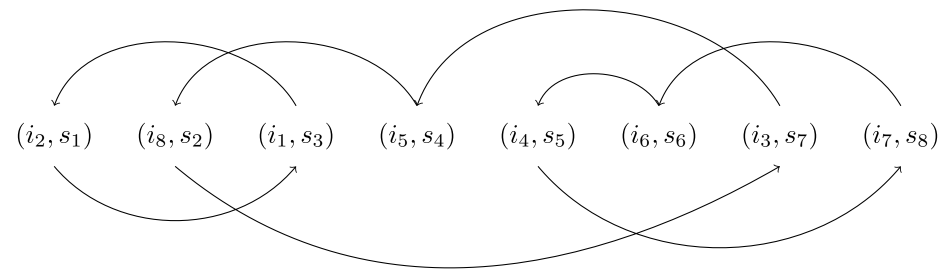
Let us start by pruning preferences and priority rankings. For example, take student 2. He is matched with s_1 in μ_I so there is no stable matching in which he will be assigned to a school he prefers to s_1. Therefore, we can remove s_6 from his preferences. At the same time, $\mu_S(i_2) = s_8$ so there is no stable assignment in which student 2 attends schools 7, 5 or 2, which are all below s_8 in i_2's preference ranking. Hence, the pruned preferences for student 2 do not include schools 6,7,5 and 2. Vice versa, we should also remove student 2 from the priority ranking of these schools.

Consider now school 1. As all the other schools in this instance, it obtains its favourite student under μ_S, so no student should be removed from its ranking according to this criterion. However, under student proposing DA, $\mu_I(s_1) = i_2$ so students 5, 7 and 6, which are all below student 2 in s_1's priority should be removed. Vice versa, we should remove school 1 from the preference list of these students.

By repeating the reasoning above for all students and schools, we get to the following pruned preferences:

$$
\begin{array}{llll}
i_1: & s_3, s_1, s_5, s_7, \bullet, \bullet, \bullet, \bullet & s_1: & i_4, i_3, i_8, i_1, i_2, \bullet, \bullet, \bullet \\
i_2: & \bullet, s_1, s_3, s_4, s_8, \bullet, \bullet, \bullet & s_2: & i_3, i_7, i_5, i_8, \bullet, \bullet, \bullet, \bullet \\
i_3: & s_7, s_4, s_3, \bullet, \bullet, s_1, s_2, \bullet & s_3: & i_7, i_5, i_8, i_3, \bullet, i_2, i_1, \bullet \\
i_4: & s_5, \bullet, s_8, \bullet, s_6, s_1, \bullet, \bullet & s_4: & i_6, \bullet, i_2, \bullet, i_3, \bullet, i_5, \bullet \\
i_5: & s_4, \bullet, s_2, s_8, s_7, s_3, s_6, \bullet & s_5: & i_8, \bullet, i_1, \bullet, i_6, i_4, \bullet, \bullet \\
i_6: & s_6, \bullet, s_5, s_7, \bullet, s_4, \bullet, \bullet & s_6: & i_5, i_4, i_7, i_6, \bullet, \bullet, \bullet, \bullet \\
i_7: & \bullet, s_8, \bullet, s_6, s_2, s_3, \bullet, \bullet & s_7: & i_1, \bullet, i_5, i_6, \bullet, i_8, i_3, \bullet \\
i_8: & s_2, \bullet, s_7, s_1, \bullet, s_3, \bullet, s_5 & s_8: & i_2, i_5, i_4, \bullet, i_7, \bullet, \bullet, \bullet
\end{array}
$$

To construct the rotation poset, we start with the graph G_{μ_I} whose vertices are pairs (i, s) such that $\mu_I(i) = s$ and i is the worst student in s's priority ranking. In this example, the vertices coincides with all student-school pairs defined by μ_I. The directed edges are drawn from a vertex (i, s) to a vertex (i', s') where s' follows s in student i's pruned preferences. The graph of student proposing DA is the following.



We can see that three rotations are exposed:
$\rho_1 = \langle (i_2, s_1), (i_1, s_3) \rangle$
$\rho_2 = \langle (i_8, s_2), (i_3, s_7), (i_5, s_4) \rangle$
$\rho_3 = \langle (i_4, s_5), (i_7, s_8), (i_6, s_6) \rangle$
After eliminating ρ_1, we obtain

$$\mu_I \setminus \rho_1 : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_1 & i_8 & i_2 & i_5 & i_4 & i_6 & i_3 & i_7 \end{array}$$

We can now prune again preferences and priority rankings. Remove s_1 (resp. s_3), and any other school above it, in the preferences of i_2 (resp. i_1). Vice versa, remove i_2 (resp. i_1), and any other student below her, in the priority ranking of s_1 (resp. s_3). Finally, check that all remaining pairs are mutually acceptable.

On the graph of $\mu_I \setminus \rho_1$, we still expose ρ_2, ρ_3 but there is no new exposed rotation. Hence, there is no direct successor of ρ_1. If we eliminate ρ_2 we get

$$\mu_I \setminus \{\rho_1, \rho_2\} : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_1 & i_5 & i_2 & i_3 & i_4 & i_6 & i_8 & i_7 \end{array}$$

and on $G_{\mu_I \setminus \{\rho_1, \rho_2\}}$ we expose a new rotation: $\rho_4 = \langle (i_2, s_3), (i_3, s_4) \rangle$. This rotation is a successor of ρ_1 and ρ_2 together.

Let us continue with eliminating ρ_4.

$$\mu_I \setminus \{\rho_1, \rho_2, \rho_4\} : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_1 & i_5 & i_3 & i_2 & i_4 & i_6 & i_8 & i_7 \end{array}$$

After pruning again, observe that on the graph of $\mu_I \setminus \{\rho_1, \rho_2, \rho_4\}$, the only exposed rotation is ρ_3. We should then eliminate it to get:

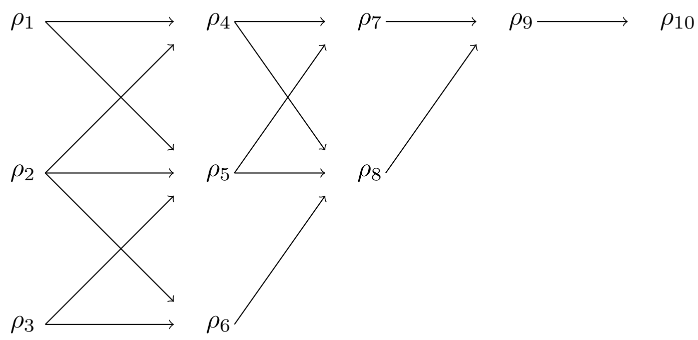
$$\mu_I \setminus \{\rho_1, \rho_2, \rho_3 \rho_4\} : \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ i_1 & i_5 & i_3 & i_2 & i_6 & i_7 & i_8 & i_4 \end{array}$$

on which graph we expose two rotations:
$\rho_5 = \langle (i_1, s_1), (i_6, s_5), (i_8, s_7) \rangle$

$\rho_6 = \langle (i_5, s_2), (i_4, s_8), (i_7, s_6) \rangle$

Observe that to expose ρ_6, which involves schools 2, 8 and 6 it is not necessary to eliminate ρ_1 which concerns only schools 1 and 3. Therefore, ρ_6 is a successor of ρ_2 and ρ_3 alone. Differently, ρ_5 is a successor of ρ_1, ρ_2, ρ_3. The other rotations that can be exposed after eliminating only ρ_4, ρ_5 or ρ_4, ρ_5, ρ_6 are, respectively, $\rho_7 = \langle (i_8, s_1), (i_3, s_3) \rangle$ and $\rho_8 = \langle (i_2, s_4), (i_5, s_8), (i_6, s_7) \rangle$. The last two rotations are $\rho_9 = \langle (i_8, s_3), (i_1, s_5), (i_5, s_7) \rangle$ and $\rho_{10} = \langle (i_3, s_1), (i_7, s_2), (i_5, s_3), (i_4, s_6) \rangle$. Where ρ_9 is exposed after eliminating ρ_7 and ρ_8, and ρ_{10} is a the successor of ρ_9. After eliminating ρ_{10} we obtain school proposing DA - i. e. $\mu_I \setminus \{\rho_j\}_{j=1}^{10} = \mu_S$.

The consequent rotation poset is



Appendix E Network flow of rotations

To find the optimal closed subset of rotations based on the method of Picard (1976), define the network flow of rotations. This is a directed weighted graph $\mathcal{G}_\Pi = (\mathcal{V}_\Pi, \mathcal{E}_\Pi)$ where the set of vertices is composed of a source vertex, $\mathcal{B}$, all the rotations and a terminal vertex, $\mathcal{T}$, i.e. $\mathcal{V}_\Pi \equiv \mathcal{B} \cup \Pi(\mathcal{I}, \mu) \cup \mathcal{T}$. There is a directed edge from $\mathcal{B}$ to each $r \in \Pi(\mathcal{I}, \mu)$ if $\omega(r) < 0$ and the weight of the edge is $|\omega(r)|$, i.e. $e = (\mathcal{B}, r) \in \mathcal{E}_\Pi$ if $\omega(r) < 0$ and $w(e) = |\omega(r)|$. There is a directed edge from $\rho \in \Pi(\mathcal{I}, \mu)$ to $r \in \Pi(\mathcal{I}, \mu)$ if r is the successor of ρ and the weight of the edge is infinity, i.e. for $\rho, r \in \Pi(\mathcal{I}, \mu)$, $e = (\rho, r) \in \mathcal{E}_\Pi$ if $\rho < r$ and $w(e) = \infty$. There is directed edge from each rotation $r \in \Pi(\mathcal{I}, \mu)$ to the terminal vertex $\mathcal{T}$ if $\omega(r) > 0$ and the weight of the edge is the weight of the rotation, i.e. $e = (r, \mathcal{T}) \in \mathcal{E}_\Pi$ if $\omega(r) > 0$ and $w(e) = \omega(r)$.

A $(\mathcal{B} - \mathcal{T})$ cut of $\mathcal{G}_\Pi$ is a partitioning of the set of vertices on the flow network into two sets, B and $\overline{B}$, such that $\mathcal{B} \in B$, $\mathcal{T} \in \overline{B}$, $B \cup \overline{B} = \mathcal{V}_\Pi$ and $B \cap \overline{B} = \emptyset$. The capacity of a $(\mathcal{B} - \mathcal{T})$ cut, denoted by $c(B, \overline{B})$, is the sum of the weights of the edges which go from the vertices in B to the vertices in $\overline{B}$. Pictorially, if the partition is a line that divides the graph of the flow network in two, the capacity is given by the sum of the weight of the edges that intersect this line in the direction from the source to the sink. We call minimum $(\mathcal{B} - \mathcal{T})$ the cut with minimal capacity among all possible cuts.

In our flow network, we can think of the rotations (the vertices) as stations that have the capacity of letting a certain flow passing through. This capacity is measured by the absolute value of the weight and the sign of the weight determines the direction the flow is sent toward: either toward the beginning $\mathcal{B}$ or toward the terminal $\mathcal{T}$. A maximum flow algorithm, for example Ford-Fulkerson algorithm (Ford & Fulkerson, 1956), would find the maximum flow this network can carry from the source vertex to the terminal vertex. Because of the duality between maximum flow and minimum cut, the maximum flow can be found with a minimum cut algorithm and this algorithm would give us the set of vertices which will carry the maximum flow. Since the capacity of each vertex in our flow network of rotations is the added value of this rotation to the social welfare function if it is eliminated, a minimum $(\mathcal{B} - \mathcal{T})$ cut in our flow network will give us the set of rotations we need to eliminate, consecutively, to find the stable matching that maximizes social welfare.