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Abstract

What does an equal sacrifice tax look like in the case of a rank-dependent social welfare function? One's tax liability evidently becomes a function of one's income and one's position in the distribution in such a case, but not much else appears to be known. (Menahem Yaari touched upon the issue in his paper "A controversial proposal concerning inequality measurement", Journal of Economic Theory 1988, but focused only on a poll tax). In this paper, we determine the properties of the equal sacrifice tax for a wide class of rank-dependent social welfare functions, and integrate the theory with that already available for the class of utilitarian social welfare functions. In an additional step, we analyze the equal sacrifice tax for a class of mixed utilitarian and rank-dependent social welfare functions, and finally we review what this synthesis has achieved.

Keywords equal sacrifice, income tax, equity.
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1. Introduction

The equal sacrifice principle in its most common form, that of equal *absolute* sacrifice, states that everyone should give up the same amount of utility when paying income taxes. When applying this principle, economists tend to assume either that everyone has the same utility function or that there is a benevolent social planner, whose welfare function is used when calculating the sacrifices to be made by taxpayers.

One function of this paper is to review the literature that has grown up around the equal sacrifice principle, which is quite an old principle.[1] This literature mostly uses a *utilitarian* framework of analysis – necessarily so, one might think, since a utility of income function is the essence of the approach. However, notwithstanding this, Young (1988) has shown that the equal sacrifice principle can be justified on the basis of non-utilitarian concepts of distributive justice, and Yaari (1988) has shown that the principle can also be articulated in terms of a rank-dependent (linear) social welfare function.

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[1] This principle was developed from the first of Adam Smith's (1776) four "canons of taxation" (that taxation should be equitable) by, *inter alia*, Mill (1848), Carver (1895), Edgeworth (1925) and Pigou (1932).

The equal sacrifice aspect of Yaari's paper has attracted very little attention. A main function of the present work is to re-examine the question of equal sacrifice taxes for rank-dependent social welfare functions, indeed to recast the appropriate criterion in terms of a more general, "mixed" class of social evaluation functions. These social evaluation functions invoke a social utility-of-income function, and they also attribute weights, giving systematically differing social importance to different people's positions in the income distribution. We re-examine in particular an old question, much addressed in the utilitarian literature: whether an equal sacrifice tax is necessarily progressive. In the new context, this proves to be a more complex question to answer, requiring simulation.

First, the simulations confirm the already well-known finding that utilitarian taxes are regressive, flat or progressive at all income levels for given parameter values. Second, the rank-dependent social welfare functions tend to generate equal sacrifice taxes that are overall progressive, but not progressive at all income levels. Third, the "mixed" social evaluation functions result in equal sacrifice taxes that may be overall progressive or regressive, but again are not necessarily so at every income level.

The paper proceeds as follows. In Section 2, we briefly review the utilitarian-based equal sacrifice literature. In Section 3, we turn to the issue of rank-dependence in the social welfare function. The equal sacrifice criterion has to be respecified in the presence of rank-dependence. We put forward the appropriate reformulation - in fact for a larger class of social welfare functions than the linear and rank-dependent ones of Yaari - a "mixed" class which includes the purely utilitarian social welfare functions as a sub-class. (The reformulated equal sacrifice criterion collapses back to the familiar one for this subclass). Section 4 concludes the paper with a summary of its main findings.

2. Utilitarian Equal Sacrifice Taxes

Equal sacrifice can mean one of three things: equal proportional, equal absolute, or equal marginal sacrifice. In common with most of the recent literature, we shall consider here mainly the absolute version of the equal sacrifice principle.[2] Thus, if $U: \mathbb{R}^+ \rightarrow \mathbb{R} \cup \{-\infty\}$ is the utility-of-income function, common to all taxpayers or imposed upon them from the outside by a social planner, and assumed strictly increasing and continuous, then the criterion for the tax schedule $t : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ to be an equal absolute sacrifice tax is that

(1) $\quad U(x) - U(x - t(x)) = u_0 \quad \forall x \geq x_0$

for some $u_o > 0$ and some $x_0 \geq 0$. If $U(0) \neq -\infty$, (1) cannot hold for all incomes $x \in \mathbb{R}^+$. The range of x for which (1) holds must be bounded away from zero in this case, else if

[2] Mill (1848, book V, chapter 2) argued for an equal absolute sacrifice, whilst Cohen Stuart (1889) argued for the proportional version of the equal sacrifice principle. Carver (1895, pp. 96-97) considered both the marginal and absolute versions of the principle, and Edgeworth (1897) favored the marginal version. The proportional version is merely the absolute version in disguise: a tax schedule t(x) engenders equal proportional sacrifice for some utility function $V: \mathbb{R}^+ \rightarrow \mathbb{R}$ if and only if the self-same schedule engenders equal absolute sacrifice for the utility function $U = \exp(V)$. See Musgrave and Musgrave (1984, chapter 11) for a neat graphical analysis of equal sacrifice criteria.

we let $x \to 0$ in (1), we would find that $t(0) > 0$, an impossible state of affairs. The quantifier $\forall x \geq x_0$ in (1) can be read as "for the incomes x of all taxpayers", since only for such persons is $t(x) > 0$ as (1) demands.[3]

What form of taxation is implied by a utilitarian equal sacrifice principle? For one thing, we can say that if U(x) is differentiable, then so too is t(x), since from (1):

(2) $t(x) = x - U^{-1}(U(x) - u_0) \quad \forall x \geq x_0$

This would in particular rule out a piecewise linear tax function. One can infer:

(3) $0 < t(x) < x \;\; \& \;\; 0 \leq t'(x) < 1 \quad \forall x \geq x_0$

from (1). The first part says that the tax is admissible on grounds of ability to pay. The second property has been termed incentive preservation (ensuring that x-t(x) increases with x). If U(x) is linear, the equal sacrifice tax is lump sum:

(4) $U(x) = ax + b, \; a > 0 \Rightarrow t(x) = u_0/a \quad \forall x \geq x_0$

(from which, $x_0 > u_0/a$ can be inferred). If U(x) is logarithmic, the tax is proportional:

(5) $U(x) = a.\ln x + b, \; a > 0 \Rightarrow t(x) = [1 - e^{-u_0/a}]x \quad \forall x \geq x_0$

In the earliest literature, it was believed that equal absolute sacrifice justified only proportional or progressive taxation.[4] Samuelson (1947, p. 227) showed that if marginal utility is sufficiently declining as income rises, equal absolute sacrifice does indeed ensure progression – in fact, strict progression:

(6) $\dfrac{-xU''(x)}{U'(x)} > 1 \;\; \forall x \geq x_0 \Rightarrow \dfrac{d}{dx}\left(\dfrac{t(x)}{x}\right) > 0 \;\; \forall x \geq x_0$

but regression is also compatible with equal absolute sacrifice: see on.

The question naturally arises, what sort of taxes can be rationalized as equal sacrifice taxes for *some* utility function U(x)? There are two strands of literature on this. One is theoretical, in which given properties of tax functions are identified with properties required of the associated utility functions, and existence theorems are developed. The other is empirical, seeking to fit actual tax systems exactly or approximately to the equal absolute sacrifice principle, using an appropriate utility function from a chosen parametric family.

Young (1987) discusses a "re-indexing property" of some tax schemes. This is where the tax t(x) gets indexed to real growth and/or inflation, becoming t*(x) = Pt(x/P) where P is the appropriate deflator. If, when t(x) is an equal sacrifice tax for some utility function U(x), so is t*(x) (with maybe a higher or lower level of sacrifice, depending on

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[3] Carver saw that the poorest would in general have to be exempted from an equal sacrifice tax. Mill argued that certain forms of income (e.g. subsistence needs and savings for retirement) should be exempted from tax. Pigou (1932, chapter 9) pointed out that an equal sacrifice tax on the better-off could finance transfers to the less well-off, also supporting the use of a bounded range in (1) to specify the equal sacrifice tax.

[4] Young (1987) cites Cohen Stuart (1889) and Edgeworth (1897) as already pointing out that this is a fallacy.

the context), for the *same* utility function U(x), then, Young shows, U(x) must take the form

(7) $U_e(x) = a\frac{x^{1-e}}{1-e}+b, \quad 0 < e \neq 1, \quad U_1(x) = a.\ln x + b, \quad a > 0$

These utilities belong to the Atkinson (1970) family, having constant inequality aversion parameter e > 0. The corresponding equal sacrifice tax function is either flat, as in (5) (for e = 1), or takes the form:

(8) $t(x) = x - [x^{1-e} - \frac{(1-e)u_0}{a}]^{1/(1-e)} \quad \forall x \geq x_0$

for $e \neq 1$. Perforce, $x_0 > (\frac{(1-e)u_0}{a})^{1/(1-e)}$ when e < 1. It is readily verified that $t^*(x) = Pt(x/P) \ \forall x \geq x_0{}^* = Px_0$ is an equal absolute sacrifice tax for the same utility function, and for the sacrifice level $u_0{}^* = P^{1-e}u_0$. It also follows from (8) that

(9) $t'(x) = 1-[1-\frac{t(x)}{x}]^e \quad \forall x \geq x_0$

from which t(x) is progressive if e > 1, proportional if e = 1 (as in (5)), and regressive (above x_0) if e < 1. Notice that the value of *b* in (7) does not affect the tax function, and the value of *a* in (7) only matters through the term $\frac{u_0}{a}$; we set *a* = 1 and *b* = 0 in all that follows.

What is the connection between the sacrifice level u_0, total tax revenue, inequality and social welfare? As in Atkinson (1970), let pre-tax social welfare be

(10a) $W_X = \int_0^\infty U(x)f(x)dx$

in case income is continuously distributed with frequency density function *f(x)*, and correspondingly,

(10b) $W_X = \frac{1}{N}\sum_{i=1}^{N} U(x_i)$

in the case of a discrete income vector $X = (x_1, x_2, \ldots x_N)$. If W_{X-T} is welfare after tax, then the welfare loss from application of the equal sacrifice tax is

(11) $\Delta W = W_X - W_{X-T} = [1-F(x_0)]u_0$

(*i.e.* a reduction of u_0 per taxpayer).[5] For the utility functions $U_e(x)$ in (7), with $e \geq 1$ (i.e. in the non-regressive case), we may write

(12) $W_X = U_e(\mu_X[1-I_X(e)]) \quad \& \quad W_{X-T} = U_e((1-g)\mu_X[1-I_{X-T}(e)])$

where μ_X is mean pre-tax income, $g\mu_X$ is the *per capita* tax payment (so that *g* is the fraction of all income taken in tax, known as the *total tax ratio*), and $I_X(e)$ and $I_{X-T}(e)$ are the pre- and post-tax Atkinson (1970) inequality indices for inequality aversion *e*. The

[5] Equation (11) does not allow for the taxation of incomes below the threshold x_o by other means. See on.

overall progression (henceforth progressivity) of the tax system can be measured in terms of these indices as:

$$(13) \quad \Pi^{BD}(e) = \frac{I_X(e) - I_{X-T}(e)}{1 - I_X(e)}$$

using the summary index of Blackorby and Donaldson (1984). Combining (11) to (13), we have[6]

$$(14a) \quad \Delta W = [1 - F(x_0)]u_0 = W_X.\{1 - [(1-g)(1+\Pi^{BD}(e))]^{1-e}\}$$

if $e \neq 1$, and

$$(14b) \quad \Delta W = [1 - F(x_0)]u_0 = -\ell n \ [(1-g)(1+\Pi^{BD}(1))]$$

when e = 1. Hence the welfare reduction is higher the greater the (required) tax yield, and lower the more progressive is the tax system. Clearly u_0, g and $\Pi^{BD}(e)$ are all interdependent (and $\Pi^{BD}(e) < 0$ for $e < 1$).

Figure 1 shows the equal sacrifice tax functions we obtained by simulation for the cases e = ½, 1, 1½, 3 using 10,000 income values x randomly drawn from a lognormal distribution, the one for which $\ell n\ x \sim N(\theta, \sigma^2)$ where $\exp(\theta) = 33.818$ is median income and the variance of logarithms is $\sigma^2 = 0.154$, and when the total tax ratio is $g = 0.15$.[7] In each panel of Figure 1, the tax level t(x) is plotted as a heavy line and the average tax rate t(x)/x is plotted as a pale line, against income x measured as a fraction or multiple of the median. In all of the simulations, we constrained average tax rates to lie between 5% and 50% in order to retain a degree of realism in the results, and continued the tax at a 5% or 50% proportionate rate outside of the range indicated for purposes of illustration. In accord with the theory, the equal sacrifice tax is regressive for e = ½, proportional for e = 1, and progressive for e = 1½ and $e = 3$. Notice that progressivity Π^{BD} (calculated for the extended equal sacrifice tax) increases as e increases.

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[6] From (12), $\mu_X[1 - I_X(e)] = U_e^{-1}(W_X)$ & $(1-g)\mu_X[1 - I_{X-T}(e)] = U_e^{-1}(W_{X-T})$, and from (13), $(1-g)[1+\Pi^{BD}(e)] = {(1-g)\mu_X[1-I_{X-T}(e)]}\Big/{\mu_X[1-I_X(e)]} = {U_e^{-1}(W_{X-T})}\Big/{U_e^{-1}(W_X)}$. The results in (14a)-(14b) follow using (7) (with a = 1, b = 0).

[7] These values of θ and σ^2 were selected for realism. The lognormal distribution they imply was found by Harrison (1981) to fit the UK distribution of gross weekly earnings in the year 1972. The average tax rate in the UK in 1972 was approximately 15% (Hutton and Lambert, 1980, pp. 902-903).

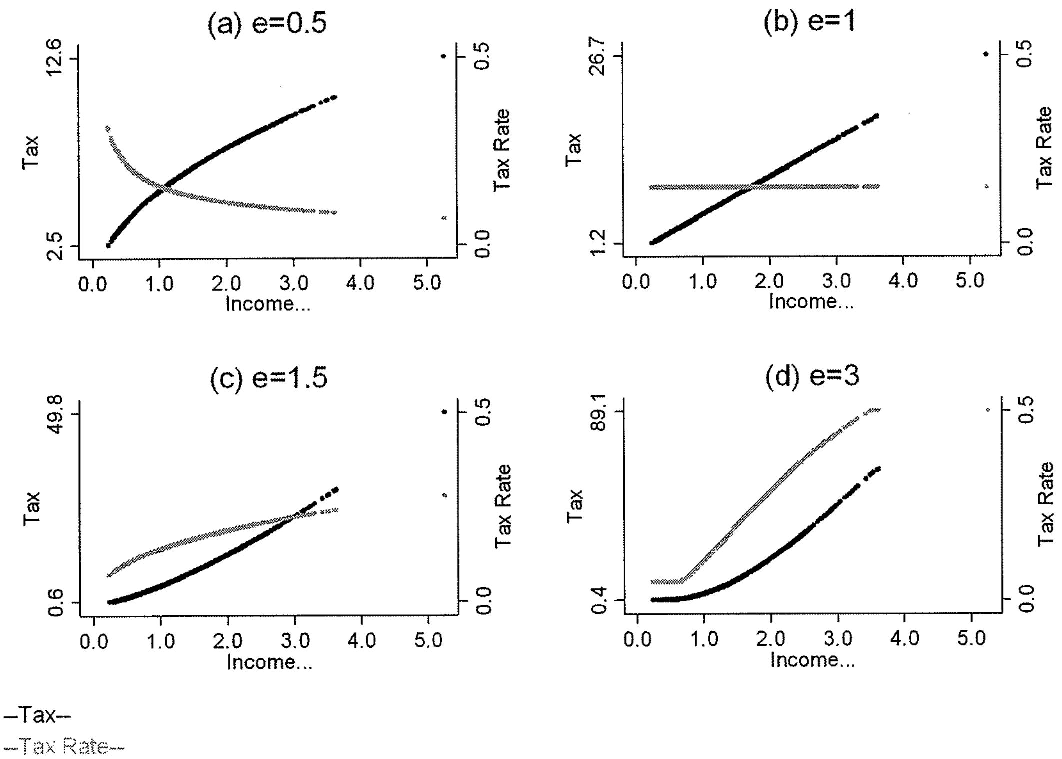

Figure 1: Utilitarian equal sacrifice taxes for (a) $e = ½$, (b) $e = 1$, (c) $e = 1½$, and (d) $e = 3$. The Blackorby-Donaldson progressivity indices are (a) $\Pi^{BD} = -0.0066$, (b) $\Pi^{BD} = 0$, (c) $\Pi^{BD} = 0.0159$ and (d) $\Pi^{BD} = 0.0759$.

Let $t_1(x)$ be the equal sacrifice tax as in (5) or (8) for inequality aversion e_1 and sacrifice level u_{01}, and let $t_2(x)$ be the corresponding equal sacrifice tax for inequality aversion $e_2 > e_1$ and sacrifice level u_{02}. It is easily shown that, whatever the two sacrifice levels u_{01} and u_{02} may be, $t_1(x)$ and $t_2(x)$ can cross *at most once*, with $t_1(.)$ crossing $t_2(.)$ from below if at all.[8] In particular, if these two taxes raise the same revenue, then the schedules must cross exactly once. This means that, for an equal absolute sacrifice income tax, post-tax inequality is unambiguously reduced when the inequality aversion of the social evaluator increases, given a constant revenue requirement.[9]

Young (1988) questions utilitarianism as the basis of the equal sacrifice principle, showing that if a tax function t(x) satisfies certain "much more primitive concepts of distributive justice" (*ibid.*, p. 322) then a utility function U exists relative to which t(x) is

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[8] From (9), $t_1'(x) = 1 - [1 - \frac{t_1(x)}{x}]^{e_1}$ and $t_2'(x) = 1 - [1 - \frac{t_2(x)}{x}]^{e_2}$. It follows that, if $t_1(y) = t_2(y)$ for some y, then $t_1'(y) < t_2'(y)$. Hence these schedules can cross at most once, as described.

[9] The single crossing property of Hemming and Keen (1983) shows this. See also Lambert (2001, p. 231), Buchholtz *et al.* (1988) and Moyes (2003) for more general consideration of the inequality-reducing properties of equal absolute sacrifice taxes.

an equal absolute sacrifice tax. An incentive-preserving schedule *t(x)* in fact only needs to (a) be *strictly monotonic* and (b) satisfy the *composition principle* (two technical properties from allocation theory) for this result.[10]

In a powerful piece of analysis, Ok (1995) shows that any continuous and strictly increasing tax function $t : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying three innocuous properties: *(i)* $t(0) = 0$, *(ii)* $0 < t(x) < x \ \forall x > 0$ and *(iii)* that the mapping $x \to x - t(x)$ be surjective, can in fact be rationalized as an equal absolute sacrifice tax for some strictly increasing and continuous utility function $U: \mathbb{R}^{++} \to \mathbb{R}$ (with x_0 set to zero in (1)), *if and only if t(x) is incentive preserving*. He makes the point that the resulting utility function may be normatively unacceptable (for example if highly convex), and goes on to prove that a *convex* tax function t(x) satisfying his postulates is an equal sacrifice tax for a *concave* utility U under a mild additional restriction.[11] Mitra and Ok (1996) show that among piecewise-linear tax schedules, essentially *only* the convex ones are equal absolute sacrifice taxes if the utility function is required to be differentiable near the origin. D'Antoni (1999) extends Mitra and Ok's analysis to show that a two-bracket piecewise linear tax function, convex or not, is an equal sacrifice tax, and moreover with respect to an entire class of concave utility functions.[12] In Mitra and Ok (1997), the piecewise linearity assumption driving this result is dispensed with, and the authors show also that some non-convex progressive tax schedules are, and some are *not*, equal absolute sacrifice taxes for concave utility functions.

The equal sacrifice criterion has also been explored empirically. Mitra and Ok (1996) demonstrated that the statutory personal income tax codes in Turkey between 1981 and 1985, and in the USA between 1988 and 1990, though progressive, were *not* equal absolute sacrifice taxes for *any* concave utility function U(.). Young (1990) successfully modelled the U.S. statutory income tax codes of 1957, 1967 and 1977 as equal absolute sacrifice taxes for isoelastic utility functions $U_e(x)$ noting, however, that "at the lower and upper ends of the distribution, the ..[isoelastic] model does not fit the data well". Gouveia and Strauss (1994) modelled *effective* U.S. income taxes annually from 1979 to 1989 as equal absolute sacrifice taxes, also for the utility function $U_e(x)$.[13]

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[10] The two questions to which technical properties (a) and (b) provide affirmative answers are these, in plain English: "If the total tax burden increases, does everyone pay more? Is the increase shared in a fair way?" (*ibid.*, p. 322). That these properties hold for an equal sacrifice tax *t(x)* is easy to see: (a) if an increase in total revenue is required while retaining equal sacrifices, plainly everyone's tax liability must rise; and (b) if a second layer of tax *s(y)* is levied on net incomes $y = x - t(x)$ to raise additional revenue, then the composite tax $s \circ t$ equalizes sacrifices if and only if the "surtax" s(y) does. Young's achievement is to obtain a converse result, that if generalized versions of (a) and (b) hold for a tax t(x), then a utility function exists relative to which t(x) equalizes sacrifices.

[11] This restriction is that t(x) be differentiable in a neighborhood of $x = 0$, with $0 < t'(0) < 1$.

[12] There is no incompatibility with Mitra and Ok's finding, because the utility functions D'Antoni comes up with all have infinitely many points of non-differentiability near the origin.

[13] Young found that the inequality aversion parameter values $e = 1.61$, $e = 1.52$ and $e = 1.72$ best fitted the 1957, 1967 and 1977 statutory codes in the US. He also obtained fits for Germany in 1984, Italy in 1987 and Japan in 1987, with $e = 1.63$, $e = 1.40$ and $e = 1.59$ respectively, but neither the US nor the UK provided a satisfactory fit in 1987. Gouveia and Strauss's best-fit values of e for the US lay between 1.72 and 1.94

In this brief perusal of the utilitarian equal sacrifice literature, the quantifier $\forall x \geq x_0$ in (1) has played a role. In the theoretical work of Young (1987, 1988), Ok (1995) and Mitra and Ok (1996, 1997), equal sacrifice is held to apply universally, i.e. x_0 is set to zero throughout, forcing the restriction U(0) = -∞ upon the respective models.[14] Yet early writers such as Carver (1895) and Pigou (1932) were content to have the equal sacrifice principle applied over a range that is bounded away from zero ($x_0 > 0$ in (1), recall footnote 3), and even Young (1990) found that his equal sacrifice model did not fit in the tails of the US income distribution. Once $x_0 > 0$ is allowed in (1), utility functions U(x) are admitted into the analysis for which U(0) is finite. Examples are the functions $U_e(x)$ for e < 1. The tax schedules corresponding to these, along with some of the schedules highlighted in the work of Mitra and Ok (1996, 1997) and D'Antoni (1999) for which $x_0 = 0$, are regressive (and one of them is shown in Figure 1(a)). In the remainder of the paper, we extend the equal sacrifice concept to rank-dependent and "mixed" social welfare functions, and here too, boundedness of the range of applicability will be appropriate.

3. Rank Dependence and Equal Sacrifice

Yaari's (1988, p. 381) proposal to "use the tax structure to retrieve the policy maker's preferences through the Principle of Equal Sacrifice" is similar in spirit to some empirical methods developed from utilitarian approaches, which we have reviewed, but there is a twist. Yaari introduces a family of social welfare functions which are *linear* in peoples' incomes and also *rank dependent*. Associated with each such welfare function is a measure which Yaari characterizes as the "equality mindedness" of the social decision maker, and this becomes an input to the determination of the tax function. In this section of the paper, we first explore the equal absolute sacrifice principle in the context of Yaari's family of social welfare functions, and then we re-examine matters using a generalized social welfare formulation in which utilitarianism and rank-dependency both figure.

3.1 Linear Social Welfare Functions and Inequality Indices.

Let $\varphi : [0,1] \rightarrow \mathbb{R}$ be increasing and twice continuously differentiable, and define an evaluation function Y_X over the pre-tax income distribution by:

$$(15a) \qquad Y_X = \int_0^\infty x\varphi'(F(x))f(x)dx$$

in case pre-tax income is continuously distributed with frequency density function *f(x)* and distribution function *F(x)*, and correspondingly, by:

[14] If U(0) is finite and $x_0 = 0$ in (1) then, as already noted, t(0) > 0 is implied. In that case, a person with zero taxable income suffers a tax burden that may eat into the very necessities of life. For J.S. Mill, for example, x = 0 connotes a person living at the subsistence level (perhaps with retirement savings set aside).

(15b) $$Y_X = \frac{1}{N}\sum_{i=1}^{N} x_i \varphi'(p_i)$$

in the case of a discrete income vector $X = (x_1, x_2, \ldots x_N)$ where, if no two of the x_i's are the same, person i's rank is $p_i = \frac{i}{N}$.[15] With certain restrictions on φ, Y_X defines a Yaari (1988) social welfare function, henceforth YSWF. Yaari associates an "equality rating" E_X with Y_X. This is defined by:

(16) $$E_X = Y_X / \mu_X$$

Changing the variable of integration in (15a), E_X can be written

(17) $$E_X = \int_0^1 L_X'(p)\varphi'(p)dp = \int_0^1 L_X(p)[-\varphi''(p)]dp$$

where $p = F(x)$ and $L_X(p)$ is the pre-tax Lorenz curve. Yaari defines the "local equality-mindedness" of the social decision-maker, at a percentile $p \in [0,1]$, as $\frac{[-\varphi''(p)]}{\varphi'(p)}$.

The restrictions on φ are most easily understood in terms of Yaari's equality rating measure.[16] First, in order that an equal distribution have an equality rating of 1, we require:

(18) $$\int_0^1 \varphi'(p)dp = 1$$

Second, in order that an extremely unequal distribution have an equality rating of 0, we require:[17]

(19) $$\varphi'(1) = 0$$

Third, in order that the evaluator be "equality-minded", we require φ to be concave:

(20) $$\varphi''(p) \le 0 \quad \forall p \in [0,1]$$

With this last restriction, $M_X = 1 - E_X$ becomes a Lorenz-consistent inequality index ((17) shows this). In fact, M_X, so defined, belongs to Mehran's (1976) "general class of linear measures of income inequality". These are weighted areas between the Lorenz curve and the line of equality, of the form:

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[15]Assuming a domain in which no two incomes are equal is not essential though it simplifies the assignation of ranks, and also incidentally makes Y_X differentiable in each x_i. Most of the analysis to follow will be demonstrated for the case in which incomes are continuously distributed.

[16] Yaari (1988) in fact defines his SWF in terms of φ'(p): he does not enumerate properties of φ.

[17] To see this, consider a discrete distribution X in which N-1 persons have ε each and person N has $N\mu_X$ - (N-1)ε. The Lorenz curve slope is $\frac{\varepsilon}{\mu_X}$ for $p \le (1-\frac{1}{N})$ and $N[1-(1-\frac{1}{N})\frac{\varepsilon}{\mu_X}]$ thereafter. From the discrete version of (17), $E_X = \frac{\varepsilon}{N\mu_X}\sum_{i=1}^{N}\varphi'(p_i) + (1-\frac{\varepsilon}{\mu_X})\varphi'(1)$. The first term is zero from the discrete version of (18). As $\varepsilon \to 0$, the distribution X approaches the extremely unequal one, and $E_X \to \varphi'(1)$.

(21) $$M_X = \int_0^1 [p - L_X(p)]k(p)dp$$

where $k(p) \geq 0 \ \forall p \in [0,1]$. In our context, k(p) = -φ"(p), hence $\int_0^1 pk(p)dp = 1$. In fact, every YSWF is of the form

(22) $$Y_X = \mu_X[1 - M_X]$$

where M_X is a Mehran index, and conversely, every Mehran index M_X defines a YSWF, as $Y_X = \mu_X[1 - M_X]$.[18,19]

The best-known Mehran indices are the Gini coefficient G, for which k(p) = 2 $\forall p$, and the extended Gini coefficient G(ν), ν > 1, of Weymark (1981), Donaldson and Weymark (1980, 1983) and Yitzhaki (1983), for which k(p) = ν(ν-1)(1-p)$^{\nu-2}$ (the case ν = 2 being that of the regular Gini). The corresponding φ-functions for the associated YSWFs are: φ(p) = 2p – p^2 for the Gini and φ(p) = 1 - (1-p)$^\nu$ for the extended Gini.

3.2 Equal Absolute Sacrifice Income Tax Functions for YSWFs

Consider a discrete pre-tax income vector $X = (x_1, x_2, ..x_N)$ and the YSWF defined as in (15b). Each dollar taken in tax from person *i* without disturbing the overall ranking of income units causes a loss of welfare of $\frac{1}{N}\varphi'(p_i)$. If the tax schedule *t(x)* is incentive-preserving, as in (3), then there is no reranking of income units in the transition from pre-tax to post-tax income; we could imagine the taxes to be subtracted sequentially, dollar by dollar from all persons whilst maintaining the ranking at every stage. Then person *i* accounts for a loss of $\frac{1}{N}t(x_i)\varphi'(p_i)$ from pre-tax welfare, and the total welfare loss is $\frac{1}{N}\sum_1^N t(x_i)\varphi'(p_i)$, call this *c > 0.* For equal sacrifices, we require:

(23) $$t(x_i)\varphi'(p_i) = c \quad i = 1,2, .. N$$

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[18] Given a weighting scheme *k(p)* ≥ 0 that defines a Mehran index M_X as in (21), where $\int_0^1 pk(p)dp = 1$ (one of Mehran's requirements), define $\varphi'(p) = \int_p^1 k(q)dq$: (18)-(20) are satisfied.

[19] Because k(p) = -φ"(p) is unrestricted except for being positive, (17) shows that in fact Lorenz dominance of one distribution over another is equivalent to dominance in terms of Yaari's equality rating for all Yaari SWFs. Further, multiplying in (17) by the mean, we obtain $Y_F = \int_0^1 GL_F(p)[-\varphi''(p)]dp$ where $GL_F(p)$ is Shorrocks's (1983) generalized Lorenz curve; whence generalized Lorenz dominance of one distribution over another is equivalent to welfare dominance for all Yaari SWFs. We thank Shlomo Yitzhaki for pointing out that these two important equivalences do not appear to have been noted in existing literature.

There is immediately a problem with this specification when $i = N$, since $\varphi'(p_N) = 0$ from (19). Note that $\varphi'(p) > 0\ \forall p \in (0,1)$ for the Gini and extended Gini weighting functions. We must therefore restrict the equal sacrifice requirement to *exclude the richest individual(s)* in the rank-dependent case:[20]

$$(24) \qquad t(x) = \frac{c}{\varphi'(F(x))} \quad \forall x : F(x) \neq 1$$

This is *not* so in the utilitarian case, and constitutes a significant distinction between the two approaches.[21] Notice, too, that for the rank-dependent case, a person's tax liability is a function of her income x through her *position* $F(x)$ in the distribution in question.

If an observed income tax function t(x) is assumed to equalize sacrifices in the Yaari sense, the equality-mindedness of the social evaluator can be inferred using (24).[22] This is the essence of Yaari's proposal. An example is provided by the case of a lump sum tax, assumed to be equal sacrifice. Let $t(x) = \tau > 0\ \forall x : F(x) \neq 1$. Then from (24), $\varphi'(p) = \frac{c}{\tau}\ \forall p \in (0,1)$, and now from (18), $c = \tau$ and zero equality-mindedness can be inferred: $[-\varphi''(p)] = 0\ \forall p$. Only if there is no concern for inequality, can a lump sum tax cause equal sacrifices all round.[23]

The equal sacrifice recipe in (24) ostensibly covers all income recipients but the very richest. However, this recipe may be inoperable at either extreme of the income distribution, or both, since admissibility on grounds of ability to pay ($0 < t(x) < x\ \forall x$) is not guaranteed by (24). Moreover, (24) is constructed assuming incentive preservation ($0 \leq t'(x) < 1\ \forall x$), and may not of itself guarantee that property (see on). Let us then relax the recipe somewhat, in fact slightly further than we needed to do in the utilitarian case. Specifically, we shall introduce upper and lower threshold income levels to delineate the region in which equal sacrifice must hold:

$$(25) \qquad t(x) = \frac{c}{\varphi'(F(x))} \quad \forall x \in [x_0, x_1]$$

for some $x_0 \geq 0$ and some x_1 such that $F(x_1) \neq 1$.

The welfare loss after application of this tax is c per taxpayer:[24]

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[20] We also exclude extremely unequal income distributions from further consideration at this point , since for such distributions the quantifier " $\forall x : F(x) \neq 1$ " simply means "for x = 0".

[21] Note, though, that Young's (1990) utilitarian equal sacrifice tax as in (8) did not fit in the upper tail of the US income distribution.

[22] From (24), $p = F(y) < 1 \Rightarrow \varphi'(p) = \frac{c}{t(y)}$, and c can be calculated using (18): $\frac{1}{c} = \int_0^\infty \frac{1}{t(x)} f(x)dx$.

[23] Yaari explains this finding in these words: "A policy maker who chooses such a tax policy (and finds it politically feasible) displays complete disregard for equality differences of income profiles. For such a policy maker, all income profiles will indeed have the same equality rating" *(ibid.*, p. 396). In the utilitarian case, it is linearity of the welfare function which yields lump sum equal sacrifice taxes (recall (4)).

[24] The analysis at this point neglects the welfare consequences which would follow from the taxation of incomes at the extremes of the distribution by non-equal-sacrifice methods – but see on.

(26) $$\Delta Y = Y_X - Y_{X-T} = [F(x_1) - F(x_0)]c = \mu_X[1 - M_X] - (1-g)\mu_X[1 - M_{X-T}]$$

Measuring progressivity in terms of pre- and post-tax Mehran indices, à la Blackorby and Donaldson (1984):

(27) $$\Pi^M = \frac{M_X - M_{X-T}}{1 - M_X}$$

we find that

(28) $$\Delta Y = [F(x_1) - F(x_0)]c = Y_X.[g - (1-g)\Pi^M]$$

The implications for welfare are similar to those in the utilitarian case: the welfare reduction is higher the greater the (required) tax yield, and lower the more progressive is the tax system. Clearly *c, g* and Π^M are all interdependent, and $\Pi^M < 0$ is possible.

In the case of the Gini-based YSWF, the equal absolute sacrifice tax is obtained by setting $\varphi'(p) = 2(1-p)$ in (25):

(29) $$t(x) = \frac{c}{2(1 - F(x))} \qquad \forall x \in [x_0, x_1]$$

and perforce in this case $x_0 > \frac{c}{2}$. Integrating in (29),

(30) $$\int_{x_0}^{x_1} t(x) f(x) dx = g\mu_X = \frac{c}{2}\int_{x_0}^{x_1} \frac{f(x)dx}{[1 - F(x)]} = \frac{c}{2}\int_{F(x_0)}^{F(x_1)} \frac{dp}{1-p} = \frac{c}{2}\ell n\left[\frac{1 - F(x_0)}{1 - F(x_1)}\right]$$

which sets an upper limit upon x_1, since the logarithm becomes unbounded as $F(x_1) \longrightarrow 1$, whereas $g \in [0,1)$.

In the case of the extended Gini, setting $\varphi'(p) = \nu(1-p)^{\nu-1}$ in (25) we have:

(31) $$t(x) = \frac{c}{\upsilon[1 - F(x)]^{\upsilon-1}} \qquad \forall x \in [x_0, x_1]$$

and hence $x_0 > \frac{c}{\upsilon}$. Integrating in (31), we obtain:

(32) $$g\mu_X = \frac{c}{\nu}\int_{F(x_0)}^{F(x_1)} \frac{dp}{(1-p)^{\nu-1}} = \frac{c}{\upsilon(\upsilon-2)}\left[\frac{1}{[1 - F(x_1)]^{\upsilon-2}} - \frac{1}{[1 - F(x_0)]^{\upsilon-2}}\right]$$

if $1 < \upsilon \neq 2$ (and (30) if $\upsilon = 2$). For $\upsilon > 2$, this sets an upper limit upon x_1.

Figure 2 shows the rank-dependent equal sacrifice tax functions for the cases υ = 1½, 2, 2½, 5 for the same lognormal income distribution as was used to construct Figure 1, and when the total tax ratio is $g = 0.15$, also as before. In each panel of the top portion of Figure 2, the tax level and average tax rate are shown against income relative to the median, and in the lower portion, these same taxes and average rates are shown as functions of position *F(x)* in the income distribution. In all cases, we terminated the equal sacrifice prescription at certain income levels, in order to keep the average tax rate within the 5% - 50% range as for the utilitarian case considered earlier, and we again continued the tax function at the highest income levels as flat, with a 50% average rate, rather than terminating it completely (failure to do this would necessitate a higher sacrifice level and greater tax burden on all other income units in order to meet the fixed yield requirement). The tax is progressive at all income levels when $\upsilon = 5$, but the average tax rate is U-shaped in all other cases, with the poorest income units experience high but declining average rates. As the lower portion of Figure 2 shows, the equal sacrifice prescription applies to more than 90% of the population for low υ; as υ increases, the tax structure becomes "highly polarized" (Yaari's words, *ibid* p. 382), in that most taxpayers are driven to one or other extreme of the permitted range for the average tax rate. Progressivity Π^M (calculated for the equal sacrifice and flat extension tax taken together) increases as υ increases.

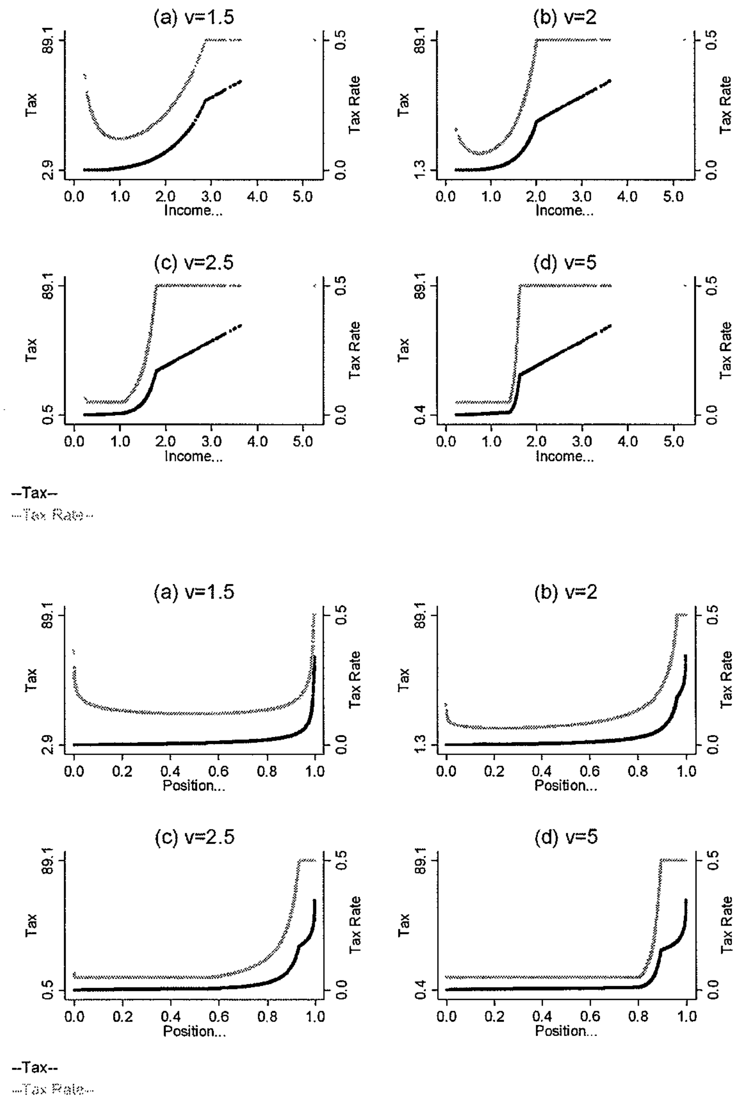

Figure 2: Rank-dependent equal sacrifice taxes for (a) $\upsilon = 1\frac{1}{2}$, (b) $\upsilon = 2$, (c) $\upsilon = 2\frac{1}{2}$, and (d) $\upsilon = 5$, as functions of income relative to the median (top) and position in the income distribution (bottom). The Mehran progressivity indices are (a) $\Pi^M = 0.0166$ (b) $\Pi^M = 0.0720$, (c) $\Pi^M = 0.0976$ and (d) $\Pi^M = 0.1111$.

The tax function for $\upsilon = 2$ is that corresponding to the Gini-based YSWF. In Figure 3, we show the sacrifice profile $\frac{1}{N}t(x_i)\varphi'(p_i)$ according to this YSWF (*i.e.* with $\varphi'(p) = 2(1-p)$) for the utilitarian equal sacrifice tax function given in (8) and displayed in Figure 1 for e = ½, 1, 1½, 3. When comparing the tax functions in Figure 1 to the one in Figure 2(b), it is clear that the two systems cannot result in the same sacrifice levels. The relationship between sacrifice and income for the utilitarian tax, when sacrifice is measured à la Yaari, has an inverse-U shape – increasing with income at low income levels and decreasing with income at higher income levels. This sacrifice profile tells us that the Yaari-sacrifice engendered by the utilitarian tax is judged to be highest around median income. None of the utilitarian equal sacrifice literature has of course been able to offer this perspective.

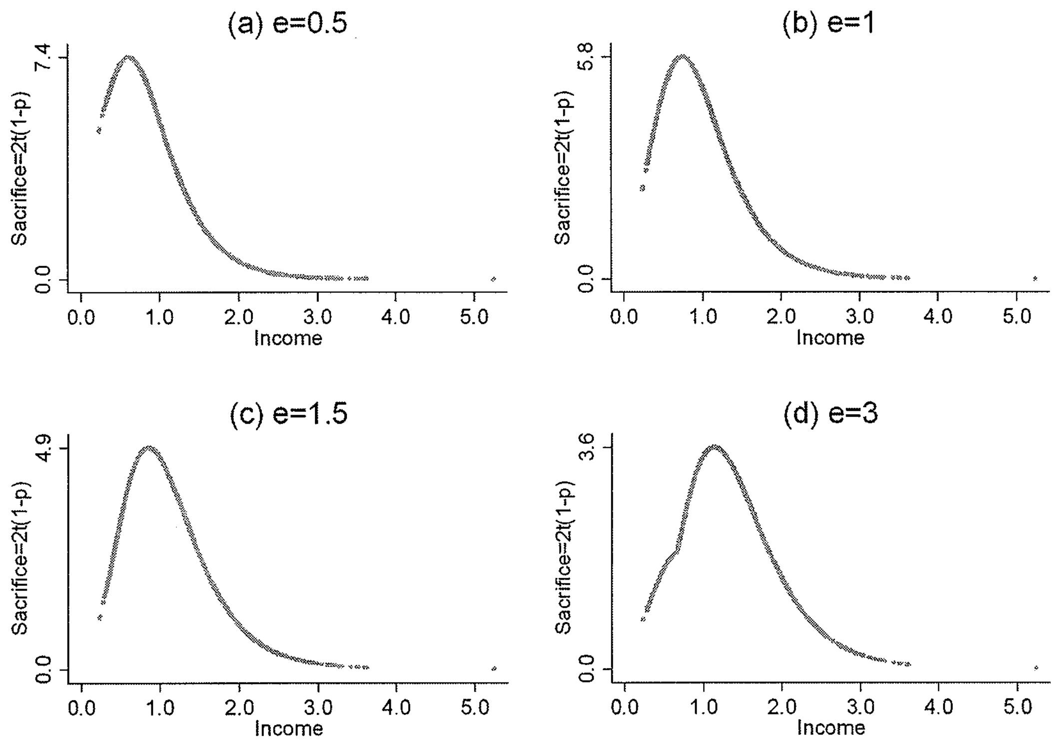

Figure 3: The sacrifice profile for the utilitarian equal sacrifice tax functions of Figure 1, with sacrifice measured according to the Gini-based YSWF.

Finally, we point out that, unlike the utilitarian case, for YSWFs there is a major difference between the equal *proportional* and equal *absolute* sacrifice rules. Each person i accounts for a welfare contribution of $x_i.\varphi'(p_i)/N$ to Y_X and a welfare loss of $t(x_i).\varphi'(p_i)/N$ as a result of taxation. Thus, only a proportional tax with the required yield, $t(x) = g.x$ $\forall x$, can engender an equal proportional sacrifice from all individuals in the Yaari framework.

3.3 Equal Sacrifice for "Mixed" Utilitarian and Rank Dependent SWFs

We now explore a doubly parametric family of "mixed" utilitarian and rank-dependent SWFs, using an approach introduced by Berrebi and Silber (1981) and further developed by Araar and Duclos (2005). Let

(33) $$Z_X(e,\upsilon) = \int_0^\infty U_e(x)\varphi_\upsilon{}'(F(x))f(x)dx$$

where $U_e(x)$ is as in (7) (with a = 1 and b = 0), and where $\varphi_\upsilon(p) = 1-(1-p)^\upsilon$, $\upsilon > 1$, is the weighting function which defines the extended Gini coefficient. Z_X defined in this way is in fact a YSWF, but one which is formulated in utility space rather than income space. Hence Z_X can equivalently be written as

(34) $$Z_X(e,\upsilon) = \overline{W}_X(e)[1-M_X(e,\upsilon)]$$

where $\overline{W}_X(e) = \int_0^\infty U_e(x)f(x)dx$ is utilitarian welfare *à la* Atkinson (1970), and $M_X(e,\upsilon)$ is an extended Mehran-type inequality index defined over utility levels. This family of evaluation functions embeds the case of a "pure" (income-denominated) YSWF:

(35a) $$e \to 0 \Rightarrow Z_X(e,\nu) \to \mu_X[1\text{-}G_X(\nu)]$$

where $G_X(\nu)$ is the extended Gini coefficient for the pre-tax income distribution X, and also embeds the case of a "pure" utilitarian (rank-*in*dependent) SWF:

(35b) $$\nu \to 1 \Rightarrow Z_X(e,\nu) \to \overline{W}_X(e) = U_e(\mu_X[1-I_X(e)])$$

where $I_X(e)$ is the Atkinson index of pre-tax income inequality.

The discrete version of (33) is $Z_X(e,\nu) = \frac{1}{N}\sum_{i=1}^{N} U_e(x_i)\varphi'(p_i)$. If the tax schedule $t(x)$ is incentive-preserving, each person i accounts for a welfare loss of $\frac{1}{N}[U_e(x_i) - U_e(x_i - t(x_i))]\varphi'(p_i)$. The condition for equal absolute sacrifice becomes

(36) $$[U_e(x_i) - U_e(x_i - t(x_i))](1-p_i)^{\upsilon-1} = \frac{c}{\upsilon} \qquad i = 1,2, .. \text{N}$$

where c is the *per capita* sacrifice, which will depend on both e and ν as well as the revenue requirement. Attending to possible problems at the two extremes of the income distribution, the "mixed" equal absolute sacrifice tax function must satisfy:

(37) $$[U_e(x) - U_e(x-t(x))] = \frac{c}{\upsilon[1-F(x)]^{\upsilon-1}} \qquad \forall x \in [x_0, x_1]$$

for some $x_0 \geq 0$ and some x_1 such that $F(x_1) \neq 1$. This tax function can be written explicitly, as:

(38a) $$\text{t(x)} = x - \left[x^{1-e} - \frac{c(1-e)}{\upsilon[1-F(x)]^{\upsilon-1}}\right]^{1/(1-e)}$$

if e $\neq$ 1, and as

(38b) $$t(x) = \left[1 - e^{-c/\upsilon[1-F(x)]^{\upsilon-1}} \right] x$$

when e = 1. Clearly the tax depends on *both* income *x and* position *F(x)* in these cases. When e < 1, the restriction

(39) $$x_0.[1 - F(x_0)]^{(\upsilon-1)/(1-e)} > \left[\frac{c(1-e)}{\upsilon} \right]^{1/1-e}$$

on x_0 is implied. Progressivity may be measured using the inequality index $M_X(e,\upsilon)$, as

(40) $$\Pi^M(e,\upsilon) = \frac{M_X(e,\upsilon) - M_{X-T}(e,\upsilon)}{1 - M_X(e,\upsilon)}$$

Figures 4a and 4b show the simulated mixed utilitarian and rank-dependent equal sacrifice tax functions for the cases *e* = ½, 1, 1½, 3 and υ = 1½, 2, 2½, 5 for the same lognormal income distribution as was used to construct Figures 1 and 2, and when the total tax ratio is *g* = 0.15, also as before. Figure 4a shows tax level and average tax rate against income *x*, and Figure 4b shows these same taxes and average rates as functions of position *F(x)* in the distribution.

It is striking that as *e* and υ increase, the interval between the two cut-offs x_0 and x_1 dictated by admissibility of the equal sacrifice prescription becomes smaller and smaller. The plots of the average tax rate are perhaps easier to understand than those of the tax level, since the cut-offs x_0 and x_1 provide 'floors' and 'ceilings' for the average tax rate profiles but bring 'cusps' into the tax level/income relationships. For e = 1 (only), which is the case in which U(x) is logarithmic, the equal sacrifice tax liability itself actually *decreases* with income, from approximately the median onwards. This is in stark contrast to the situation in the non-mixed (pure rank-dependent) setting, where taxes must increase with income (see (24), also Yaari, *ibid.*, p. 395). In the utilitarian setting, the case e = 1 generated a proportional tax.

Table 1 specifies the progressivity measures $\Pi^M(e,\upsilon)$ for the taxes illustrated in Figures 4a and 4b. Progressivity increases in absolute value as either e or υ increases (with the other held constant). If e = 1 the tax is everywhere regressive, whilst if e > 1 the tax is everywhere progressive. In the utilitarian case, it was e = ½ that resulted in a regressive tax. Here, for e = ½ the tax has regressive and progressive portions, but is overall progressive.

	υ = 1½	υ = 2	υ = 2½	υ = 5
e = ½	0.0348	0.0836	0.0944	0.0628
e = 1	-0.0488	-0.1061	-0.1641	-0.3450
e = 1½	0.0750	0.1019	0.1078	0.1145
e = 3	0.1038	0.1116	0.1136	0.1164

Table 1: Progressivity $\Pi^M(e,\upsilon)$ of the mixed equal sacrifice income tax for *e* = ½, 1, 1½, 3 and υ = 1½, 2, 2½ and 5.

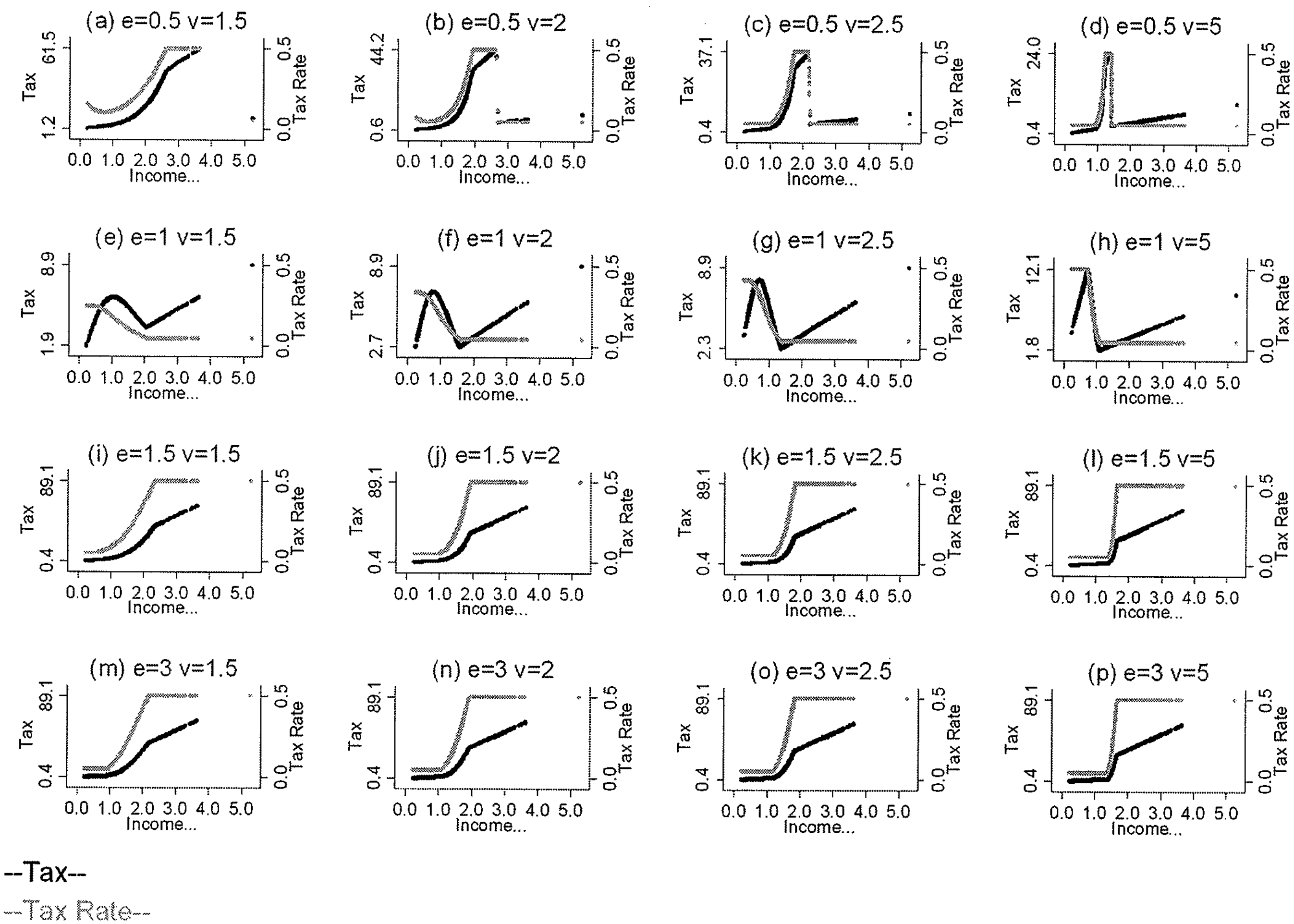

Figure 4a: Mixed utilitarian and rank-dependent equal sacrifice taxes and average tax rates in terms of the parameters *e* and υ, shown as functions of income.

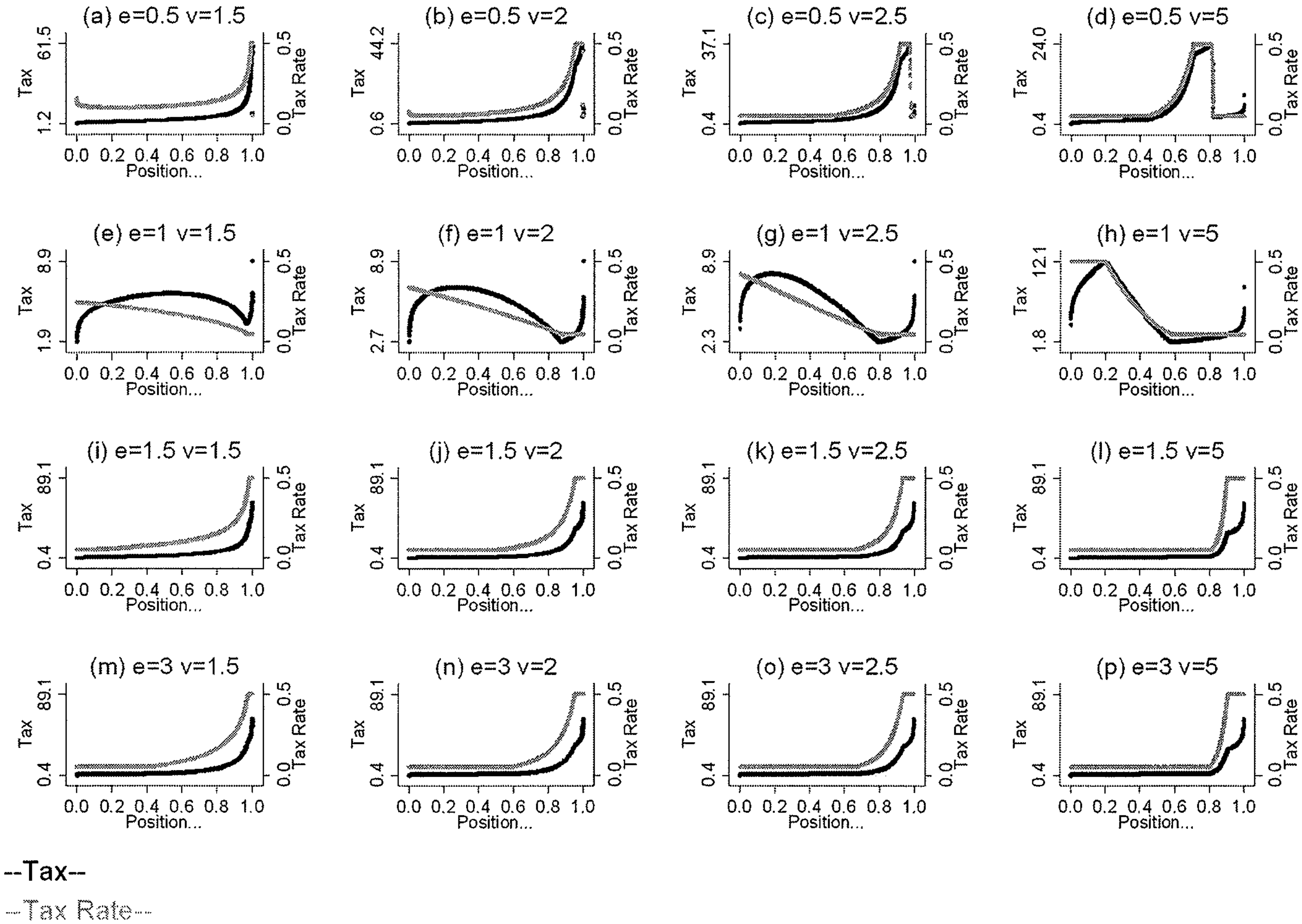

Figure 4b: Mixed utilitarian and rank-dependent equal sacrifice taxes and average tax rates in terms of the parameters e and υ, shown as functions of position.

Figure 5 contrasts the equal sacrifice tax functions that arise in the utilitarian case (when $e = ½, 1, 1½, 3$) with those that arise in the Gini-based mixed case (when $\upsilon = 2$ and also $e = ½, 1, 1½, 3$). There are considerable differences between the two tax functions for $e = ½$ and $e = 1$, but as e increases further they become more similar. This result is driven to some extent by our chosen constraint on average tax rates, that they should lie between 10% and 50%. Clearly, the rank-dependent perspective allows us to characterize a range of potentially interesting and unfamiliar tax schedules as equal sacrifice taxes.

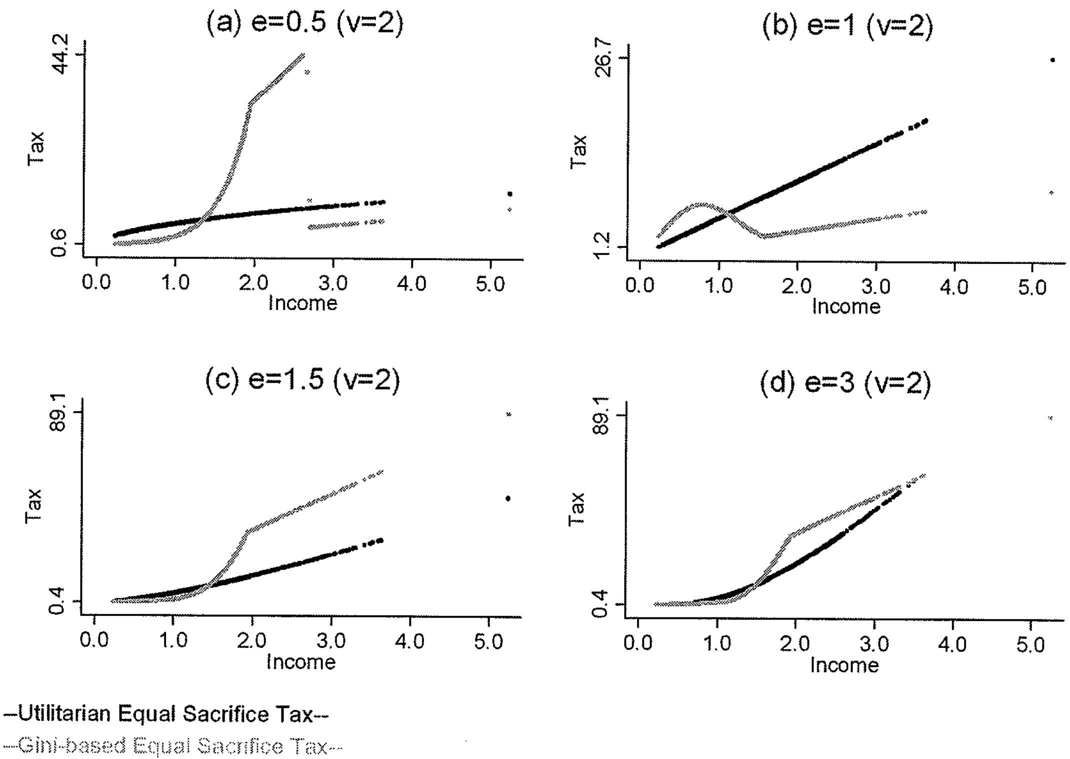

Figure 5: Comparison of utilitarian equal sacrifice tax functions and Gini-based mixed equal sacrifice tax functions, for inequality aversion values $e = ½, 1, 1½$ and 3.

Conclusions

This paper summarizes previous literature on equal sacrifice taxes. Most of that literature has focused on the class of utilitarian social welfare functions. Yaari (1988), in contrast, discusses the equal sacrifice principle in the framework of a rank-dependent social welfare function. While the properties of utilitarian equal sacrifice taxes are well-studied by now, the properties of rank-dependent equal sacrifice taxes have not yet been investigated in depth.

We compare the properties of the equal sacrifice taxes for rank-dependent and "mixed" social welfare functions to those generated for utilitarian social welfare functions. Previously studied utilitarian approaches generate equal sacrifice taxes that are progressive, flat or regressive at all levels of income. In comparison, the rank-dependent equal sacrifice taxes are overall progressive, but in general the average tax rate is a U-shaped function of position. The "mixed" social welfare functions, which are concave functions of individuals' income levels and also are functions of their positions in the income distribution, result in equal sacrifice taxes which, in our simulations, are progressive at all income levels for $e > 1$ and regressive at all income levels for $e = 1$. Only in the case $e = \frac{1}{2}$ did we find the average tax rate to be a non-monotonic (U-shaped) function of income. The Gini-based mixed equal sacrifice taxes differ markedly from the corresponding utilitarian ones for low values of inequality aversion e, but become quite similar as inequality aversion is increased. Sacrifice for the utilitarian taxes, as judged by the Gini-based YSWF, is maximal at around median income.

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